Løsningsforslag eksamen i MA1202 3. juni 2005

Oppg 1

a) $t=0$ $\quad A_0 = \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$ $\quad$ rank$(A_0) = 3$

$t \neq 0$ $\quad A_t \sim \begin{pmatrix} 1&0&0&0\\ 0&1&t&0\\ 0&0&1&t\\ 0&1&0&\frac{1}{t} \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0\\ 0&1&t&0\\ 0&0&1&t\\ 0&0&-t&\frac{1}{t} \end{pmatrix} \sim \begin{pmatrix} 1&0&0&0\\ 0&1&t&0\\ 0&0&1&t\\ 0&0&0&t^2+\frac{1}{t} \end{pmatrix}$

$\det(A_t) = 0$ når $t^2 + \frac{1}{t} = 0 \iff t^3 = -1 \iff t = -1$

for $t \neq 0$ og $t \neq -1$ rang$(A_t) = 4$, $\det(A_t) \neq 0$

for $t = -1$ $\quad A_{-1} = \begin{pmatrix} 1&0&0&0\\ 0&1&-1&0\\ 0&0&1&-1\\ 0&0&0&0 \end{pmatrix}$ $\quad$ rang$(A_{-1}) = 3$

b) 1. $t \neq 0$ og $t \neq -1$ rang$(A_t) = 4$ systemet har 1 og bare 1 løsning

2. $t = 0$ $\quad A_0 x = \begin{pmatrix} 1\\1\\0\\-1 \end{pmatrix}$ siden $\begin{pmatrix} 1\\1\\0\\-1 \end{pmatrix} \notin$ søylerom A_0

fins det ingen løsning til $A_0 x = \begin{pmatrix} 1\\1\\0\\-1 \end{pmatrix}$

3. $t = -1$ $\quad \begin{pmatrix} -1&0&0&0\\ 0&1&-1&0\\ 0&0&1&-1\\ 0&-1&0&1 \end{pmatrix} x = \begin{pmatrix} 1\\2\\1\\-3 \end{pmatrix}$

$$\begin{array}{cccc} \underbrace{}_{a_1} & \underbrace{}_{a_2} & \underbrace{}_{a_3} & \underbrace{}_{a_4} \end{array}$$

$\begin{pmatrix} 1\\2\\1\\-3 \end{pmatrix} = -a_1 + a_2 - a_3 - 2a_4 \Rightarrow \begin{pmatrix} 1\\2\\1\\-3 \end{pmatrix} \in$ søylerom A_{-1}

(*) $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -3 \end{pmatrix} \in$ Søylerom A_{-1} $\Rightarrow$ det fins minst en løsning av $A_{-1}x = b_{-1}$ (2)

(**) rank $(A_{-1}) = 3 < 4$ $\Rightarrow$ det fins vendelig mange løsninger av $A_{-1}x = 0$

(*) (**) $\Rightarrow$ $A_{-1}x = b_{-1}$ har vendelig mange løsninger (side 249 Anton Rorres)

Oppgave 2

a) $p(x) \in P_2$ $p(x) = a_0 + a_1 x + a_2 x^2$ $T_1(p) = a_0 x + a_1 x^2 + a_2 x^3$

$$(T_2 \circ T_1)(p) = T_2(T_1(p)) = a_0 + 2a_1 x + 3a_2 x^2$$

nå gitt $q(x) = b_0 + b_1 x + b_2 x^2$, vil vi finne $p(x) = a_0 + a_1 x + a_2 x^2$

s.a. $p(x) = (T_2 \circ T_1)^{-1}(q)$.

$$p = (T_2 \circ T_1)^{-1}(q) \Leftrightarrow (T_2 \circ T_1)p = q$$

$$\Leftrightarrow a_0 + 2a_1 x + 3a_2 x^2 = b_0 + b_1 x + b_2 x^2$$

$$\Leftrightarrow \begin{matrix} a_0 = b_0 \\ 2a_1 = b_1 \\ 3a_2 = b_2 \end{matrix} \Leftrightarrow \begin{matrix} a_0 = b_0 \\ a_1 = \frac{b_1}{2} \\ a_2 = \frac{b_2}{3} \end{matrix}$$

d.v.s. $(T_2 \circ T_1)^{-1}(q) = (T_2 \circ T_1)^{-1}(b_0 + b_1 x + b_2 x^2) = b_0 + \frac{b_1}{2} x + \frac{b_2}{3} x^2$

b) $B = \{1, (x-1), (x-1)^2\}$

$$(T_2 \circ T_1)(1) = 1 \qquad [(T_2 \circ T_1)(1)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(T_2 \circ T_1)(x-1) = 2x-1 \qquad [(T_2 \circ T_1)(x-1)]_B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$= 2(x-1)+1$$

$$(T_2 \circ T_1)((x-1)^2) = (x-1)^2 + 2x(x-1) \qquad [(T_2 \circ T_1)(x-1)^2]_B = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$
$$= 3(x-1)^2 + 2(x-1)$$

$$[(T_2 \circ T_1)]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

c) $p \in \mathbb{P}_3 \quad p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$T_2(p) = a_1 + 2a_2 x + 3a_3 x^2$$

Teor 8.2.3 s. 379 Anton & Rorres

$$\dim \mathbb{P}_3 = 4 = \dim \ker(T_2) + \dim R(T_2)$$

nå $\ker(T_2) = \{ q \in \mathbb{P}_3 : \frac{dq}{dx} = 0 \} = \{ \text{konstanter} \}$
$= \text{span}\{1\}$

$\Rightarrow \dim \ker(T_2) = 1$
$\text{nullity}(T_2) = 1$

$4 = 1 + \text{rang}(T_2) \quad \Rightarrow \text{rang}(T_2) = 3$

Oppgave 3

a) Egenverdiene til A:

$$\det\begin{pmatrix} \lambda+2 & -1 & 0 & 0 \\ -1 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda+2 & -1 \\ 0 & 0 & -1 & \lambda+2 \end{pmatrix} = (\lambda+2)(\lambda+2)\left((\lambda+2)^2-1\right) + $$

$$+1\cdot\left(-1\cdot\left((\lambda+2)^2-1\right)\right) =$$

$$= (\lambda+2)^2\left((\lambda+2)^2-1\right) - \left((\lambda+2)^2-1\right)$$

$$= \left(\underbrace{(\lambda+2)^2-1}\right)\left((\lambda+2)^2-1\right)$$

$$\lambda^2+4\lambda+4-1=0$$

$$\lambda^2+4\lambda+3=0$$

$$\lambda_{12} = \begin{cases} -3 \\ -1 \end{cases}$$

Egenvektorer til A:

$$\lambda=-1 \quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

løsning $x = \begin{pmatrix} t \\ t \\ s \\ s \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$$\text{egenrom} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\} = \text{span}\underbrace{\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right\}}_{\text{ortonormal basis}}$$

$$\lambda=-3 \quad \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Cosming $$x = \begin{pmatrix} r \\ -r \\ p \\ -p \end{pmatrix} = r \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + p \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

eigenrau $$= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\text{ortonormal basis}} \right\}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad P^T P = I$$

$$P = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \qquad A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \qquad P^T = P$$

$$P^T A P = P A P = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} Q\tilde{A}Q & 0 \\ 0 & Q\tilde{A}Q \end{pmatrix}$$

$$Q\tilde{A}Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

b) $A = PDP \qquad P^TP = I$

$$y' = Ay \iff y' = PDPy \qquad z := Py$$
$$y(0) = y_0 \qquad\qquad y(0) = y_0$$

$$\dot{z} = P\dot{y} = Dz$$
$$z(0) = Py(0)$$

$$z(0) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-2}{\sqrt{2}} \\ 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$z(t) = \begin{pmatrix} e^{-t} \cdot 0 \\ \frac{-2}{\sqrt{2}} e^{-3t} \\ 0 \\ \frac{2}{\sqrt{2}} e^{-3t} \end{pmatrix} = \frac{2}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

$$y(t) = Pz(t) = \frac{2}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} e^{-3t} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} e^{-3t}$$

Oppgave 4

$$\tau_1 = \frac{\tau_2 + \tau_4}{4}$$
$$\tau_2 = \frac{\tau_1 + \tau_3 + \tau_5}{4}$$
$$\tau_3 = \frac{\tau_2 + \tau_6}{4}$$
$$\tau_4 = \frac{\tau_1 + \tau_5 + 1}{4}$$

$$\tau_5 = \frac{\tau_4 + \tau_2 + \tau_6 + \tau_7}{4}$$

$$\tau_6 = \frac{\tau_5 + \tau_3 + 1}{4}$$

$$\tau_7 = \frac{1 + \tau_5 + 1 + 1}{4}$$

ganger begge sider av alle ligninger med 4 og setter alle ukjente på venstre side og

$$\underbrace{\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}}_{b}$$

Oppgave 5

Anta A $n \times n$ kompleks, skjevhermitsk d.v.s. $A^* = -A$ $\quad A^* = \bar{A}^T$

$Ax = \lambda x$ $\quad \lambda$ eigenverdi x egenvektor, anta $x^* x = 1$

da $\quad x^* A x = \lambda$

og $\quad \bar{\lambda} = \overline{x^* A x} = x^* A^* x = -x^* A x = -\lambda$

λ er et komplekstall d.v.s. $\lambda = a + ib$

og $\bar{\lambda} = a - ib$

$\bar{\lambda} = -\lambda \iff a \cancel{-ib} = -a \cancel{-ib} \Rightarrow 2a = 0 \Rightarrow a = 0$

$\lambda = ib$ ren immaginær.