Eksamen i MA1202
30. mai 2007

Opg 1

a)

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -2 & 1 & -2 \\ 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) = "antall ledende enere i redusert trappeform" $\underline{\underline{= 2}}$

Basis for kolonnerommet: $\underline{\underline{\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}}}$

b) null(A) + rank(A) = 4

$\Rightarrow$ null(A) = 4 − 2 $\underline{\underline{= 2}}$

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -2 & 1 & -2 \\ 1 & 3 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ vektoren $\begin{bmatrix} -3 \\ 2 \\ -2 \\ -1 \end{bmatrix}$ er ikke i nullrommet

$\underline{\underline{\text{til } A.}}$

Opg 2

a) $\begin{bmatrix} a+b & b+d \\ -b-d & a+c \end{bmatrix} \in V$, så V er lukket under addisjon

$\begin{bmatrix} ra & rb \\ -rb & ra \end{bmatrix} \in V$, så V er lukket under skalar multiplikasjon

$\Rightarrow$ V er et underrom av $M_{2\times 2}(\mathbb{R})$

$\Rightarrow$ V er et vektorrom

b) La $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

Ser at $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

så $\operatorname{span} B = V$

La $r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} r & s \\ -s & r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow r = 0 \;\&\; s = 0$

$\Rightarrow$ B er lineært uavhengig

$\Rightarrow$ B er en basis for V.

$\Rightarrow \dim_{\mathbb{R}} V = 2$

Opg 3

a)

$$[T]_{B'}^{B} = \left[[T(1)]_{B'} \;\vdots\; [T(x)]_{B'} \;\vdots\; [T(x^2)]_{B'} \right]$$

$$= \left[[0]_{B'} \;\vdots\; [1]_{B'} \;\vdots\; [2x+2]_{B'} \right]$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$[T(2x^2-4)]_{B'} = [T]_{B'}^{B} \, [2x^2-4]_B$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow T(2x^2-4) = 4 \cdot (1+x) + 0 \cdot (1-x)$$

$$= 4 + 4x$$

b)

$$T(1) = 0 = T(0)$$

$\Rightarrow$ T er ikke 1-1

$$R(T) = \{ T(a_0 + a_1 x + a_2 x^2) \mid a_i \in \mathbb{R} \}$$

$$= \left\{ \frac{d\left(a_0 + a_1 + a_2 + (a_1 + 2a_2)x + a_2 x^2\right)}{dx} \;\middle|\; a_i \in \mathbb{R} \right\}$$

$$= \{ a_1 + 2a_2 + 2a_2 x \mid a_i \in \mathbb{R} \} = P_1$$

Opg 4

a) $A = \begin{bmatrix} -1 & 0 & 0 \\ -5 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 5 & \lambda+2 & -3 \\ 5 & 1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda+1)(\lambda+2)(\lambda-2) + 3(\lambda+1)$$

$$= (\lambda+1)\,(\lambda^2-4+3) = (\lambda+1)(\lambda^2-1)$$

$$= (\lambda+1)(\lambda+1)(\lambda-1)$$

Egenverdier : $\underline{\lambda_1 = -1 \;\&\; \lambda_2 = 1}$

$$\lambda_1 I - A = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 1 & -3 \\ 5 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 3x_3 - 5x_1$

Egenrommet tilhørende λ_1 = $\underline{\text{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right\}}$

$$\lambda_2 I - A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & -3 \\ 5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = x_3 \quad x_1 = 0$

Egenrommet tilhørende λ_2 = $\underline{\text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}}$

b) $P = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ diagonaliserer A, &

$$P^{-1}AP = \underline{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = D$$

$$A^{97} = (PDP^{-1})^{97} = PD^{97}P^{-1} = PDP^{-1} = \underline{A}$$

$$A^{136} = (PDP^{-1})^{136} = PD^{136}P^{-1} = PIP^{-1} = \underline{I}$$

c)

$$\begin{bmatrix} -1 & 0 & 0 \\ -5 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix} \begin{bmatrix} g(x) \\ f(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} g'(x) \\ f'(x) \\ h'(x) \end{bmatrix}$$

$$A \underline{x} = \underline{x}'$$

$$PDP^{-1} \underline{x} = \underline{x}'$$

$$DP^{-1} \underline{x} = P^{-1} \underline{x}'$$

La $\underline{y} = P^{-1} \underline{x}$, $\underline{y} = \begin{bmatrix} a(x) \\ b(x) \\ c(x) \end{bmatrix}$

$$D \underline{y} = \underline{y}'$$

$$-a(x) = a'(x) \Rightarrow a(x) = r e^{-x}$$
$$-b(x) = b'(x) \Rightarrow b(x) = s e^{-x}$$
$$c(x) = c'(x) \Rightarrow c(x) = t e^{x}$$

$$\underline{x} = P\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} re^{-x} \\ se^{-x} \\ te^{x} \end{bmatrix}$$

$$= \begin{bmatrix} re^{-x} \\ -5re^{-x} + 3se^{-x} + te^{x} \\ se^{-x} + te^{x} \end{bmatrix}$$

$$-1 = g(0) = r$$
$$1 = f(0) = -5r + 3s + t$$
$$2 = h(0) = s + t$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -5 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 3 & 1 & -4 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{aligned} g(x) &= -e^{-x} \\ f(x) &= -4e^{-x} + 5e^{x} \\ h(x) &= -3e^{-x} + 5e^{x} \end{aligned}$$

Opg 5

Anta $\langle \underline{u}, \underline{u} \rangle = a + ib$, $a, b \in \mathbb{R}$

Siden $\langle -, - \rangle$ er et kompleks indreprodukt,

så er $\langle \underline{a}, \underline{b} \rangle = \overline{\langle \underline{b}, \underline{a} \rangle}$

$$\Rightarrow \quad a + ib = \langle \underline{u}, \underline{u} \rangle = \overline{\langle \underline{u}, \underline{u} \rangle} = a - ib$$

$$\Rightarrow b = 0 \quad \Rightarrow \quad \underline{\underline{\langle \underline{u}, \underline{u} \rangle = a \in \mathbb{R}.}}$$