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TALLTEORI (MA1301)

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① We give three proofs that $\frac{6^n - 1}{5}$ is an integer, $n = 1, 2, 3, \dots$:

- (i) INDUCTION

  Claim: $5 \mid 6^n - 1 \quad (n = 1, 2, 3, \dots)$

  1°) <u>n=1</u> $\quad 6^1 - 1 = 5$ valid.

  "$n = k$" 2°) <u>Induction hypothesis</u>: $5 \mid 6^k - 1$, say $6^k - 1 = 5N_k$.

  "$n = k+1$" 3°) $6^{k+1} - 1 = 6 \cdot 6^k - 1 = 5 \cdot 6^k + 6^k - 1$

  $\overset{\text{IND.}}{\underset{\text{HYP.}}{=}} 5[6^k + N_k]$, i.e. $5 \mid 6^{k+1} - 1$

  The Principle of Induction guarantees that the claim holds for each $n = 1, 2, 3, \dots$ □

- (ii) $6 \equiv 1 \pmod 5$

  $6^n \equiv 1^n = 1 \pmod 5$ or $5 \mid 6^n - 1$. □

- (iii) $6^n - 1 = \underbrace{(6-1)}_{5}\underbrace{(1 + 6 + 6^2 + \dots + 6^{n-1})}_{\text{INTEGER}}$

  (The sum of a geometric series).

  The factor 5 is displayed! □

<u>Remark</u>: $1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}$

(2) $710x + 68y = 6 \iff$

$355x + 34y = 3$

Euclid's algorithm:

I

$355 = 10 \cdot 34 + 15$

$34 = 2 \cdot 15 + 4$

$15 = 3 \cdot 4 + 3$

$4 = 1 \cdot 3 + \underline{1}$

$3 = 3 \cdot 1$

Reversed:

II $1 = 4 - 3 =$

$4 - (15 - 3 \cdot 4)$

$= -15 + 4 \cdot 4$

$= -15 + 4(34 - 2 \cdot 15)$

$= 4 \cdot 34 - 9 \cdot 15$

$= 4 \cdot 34 - 9(355 - 10 \cdot 34)$

$= -9 \cdot 355 + 94 \cdot 34$

$355(-9) + 34 \cdot 94 = 1$

III $\boxed{355(-27) + 34 \cdot 282 = 3}$

Solutions:

IV $\begin{cases} x = -27 + 34t \\ y = 282 - 355t \end{cases}$

Ex.:
$x = 7$
$y = -73$

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(3) Antithesis $4n^3 = m^3$; $\gcd(m,n) = 1$

upon division of common factors

$2 | m^3 \implies 2 | m$ Hence $m = 2\mu$

$4n^3 = 8\mu^3$, $n^3 = 2\mu^3$ Again $2 | n$

Contradiction: $\gcd(m,n) \geq 2$.

Hence the antithesis is false and $\sqrt[3]{4}$ is irrational.

(4) The number $3n+2$ $(n \geq 0)$ has factors of the type

$$\begin{cases} 3k & \text{impossible,} \\ 3k+1, & \\ 3k+2. & \end{cases}$$

GROUPING MODULO 3.

The product of numbers of the type $3k+1$ is again of the same type:

$$(3k+1)(3l+1) = 3[3kl+k+l]+1$$
$$= 3m+1$$

It follows that $3n+2$ cannot have prime factors of <u>only</u> the type $3k+1$. Thus there must be at least one prime factor of the form $3k+2$ (including the possibility that the number itself was a prime).

ADDENDUM. Using

$$3(5 \cdot 7 \cdot 11 \cdots p_n) + 2$$

one may conclude that there are infinitely many primes of the form $3n+2$. (This is a special case of Dirichlet's theorem about primes in arithmetic progressions.)