Chebyshev-polynomials

The Chebyshev-polynomials are defined by

$$T_n(x) = \cos\left[n \arccos x\right], \qquad x \in [-1,1], \quad n = 0,1,2,\dots$$

They satisfy the following recursion formula (see Problem set 4)

$$\begin{aligned} &T_0(x) = 1, \qquad T_1(x) = x \\ &T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2,3,\dots \end{aligned}$$

The Chebyshev-polynomials have the following properties:

$$|T_n(x)| \le 1 \quad \text{and} \quad T_n(\hat{x}_i) = (-1)^i, \quad \hat{x}_i = \cos\left(\frac{i\pi}{n}\right), \quad i = 0,1,\dots,n. \tag{1}$$

$$T_n(x_i) = 0, \qquad \text{for} \quad x_i = \cos\left(\frac{2i+1}{2n}\pi\right), \quad i = 0,1,\dots,n-1. \tag{2}$$

$$T_n(x) = 2^{n-1}x^n + \cdots \tag{3}$$

Let

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x) \in \tilde{\mathbb{P}}_n$$

where $\tilde{\mathbb{P}}_n = \{p \in \mathbb{P}_n, \quad p(x) = x^n + a_{n-1}x^n + \cdots + a_0\}$.

Theorem (*min/max property*): The polynomials $\tilde{T}_n(x)$ satisfy

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} \left|\tilde{T}_n(x)\right| \le \max_{x\in[-1,1]} |q(x)|, \quad \text{for all} \quad q \in \tilde{\mathbb{P}}_n.$$

Proof: Assume that $q \in \tilde{\mathbb{P}}_n$ satisfy

$$\max_{x\in[-1,1]} |q(x)| < \frac{1}{2^{n-1}}.$$

Let $r = \tilde{T}_n - q$, such that $r \in \mathbb{P}_{n-1}$. This means

$$r\left(\hat{x}_i\right) < 0 \quad \text{for } i \text{ odd}, \qquad r\left(\hat{x}_i\right) > 0 \quad \text{for } i \text{ even}, \quad i = 0,1,\dots,n,$$

and r must have at least one root in each of the intervals $(\hat{x}_i, \hat{x}_{i+1})$, $i = 0,1,\cdots,n-1$. However, the polynomial r is of degree $n-1$ and has at least n zeros. This shows that the assumption is wrong.

□

Combined with Theorem 1 p.156 in C&K this leads to

Corrolary: If $p(x) \in \tilde{\mathbb{P}}_n$ where the interpolation nodes are the zeros of $T_{n+1}(x)$, we have

$$\max_{x\in[-1,1]} |f(x) - p(x)| \le \frac{1}{2^n(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)|, \qquad f \in C^{n+1}[-1,1].$$