MA2501 Numerical methods

Fixed-point iteration

A *fixed point* for a given function g is a number r such that $r = g(r)$. A fixed-point iteration scheme is given by

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots.$$

We are interested to know:

- Does a fixed-point exist?
- Is the fixed-point unique?
- If there is a fixed-point, will the sequence $\{x_n\}$ converge to this number?

The following theorem will give us the answer:

Theorem 1 *Assume that*

i) $g \in C[a,b]$ *and* $g(x) \in [a,b]$ *for all* $x \in [a,b]$.

ii) $g'(x)$ *exists for all* $x \in (a,b)$ *and there is a positive konstant* $\rho < 1$ *such that*

$$|g'(x)| \leq \rho < 1, \qquad \textit{for all } x \in (a,b).$$

1. *If assumption i) is satisfied there exists at least one fixed-point in* $[a,b]$.
2. *If assumption ii) is satisfied in addition to i), the fixed-point is unique, and the fixed-point iterations will converge to this number for all start-values* $x_0 \in [a,b]$.

Proof: Let us first show the existence of a fixed-point. If $g(a) = a$ or $g(b) = b$ we have found it. Let's consider the situation when neither a or b is a fixed-point. Then we must have $g(a) > a$ and $g(b) < b$. Define a function $h(x) = g(x) - x$. Then

$$h(a) = g(a) - a > 0 \qquad \text{and} \qquad h(b) = g(b) - b < 0.$$

Since h is continuous on $[a, b]$, and h changes sign, there must exist an $r \in (a, b)$ such that $h(r) = 0$. r is then a fixed-point for g.

We show the uniqueness of the fixed-point by showing that two different fixed-points is impossible when *ii)* is satisfied. Let r and q both be fixed-points of g in the interval $[a, b]$, and assume that $r \neq q$. The mean value theorem says that there exists a ξ between r and q such that

$$r - q = g(r) - g(q) = g'(\xi)(r - q).$$

Since r and q are both in $[a, b]$ then $\xi \in (a, b)$. If *ii)* is satisfied we have

$$|r - q| = |g(r) - g(q)| = |g'(\xi)||r - q| < |r - q|,$$

which is impossible. Hence, the fixed-point is unique.

Let us consider the fixed-point iterations. Assume that $x_0 \in [a, b]$. Since $g(x) \in [a, b]$ for all $x \in [a, b]$, we will have $x_n \in [a, b]$, $n = 0, 1, 2, \ldots$. Then we have

$$|r - x_n| = |g(r) - g(x_{n-1})| = |g'(\xi_n)||r - x_{n-1}| < \rho|r - x_{n-1}| < \rho^n|r - x_0|.$$

Since $\rho < 1$ we have that

$$\lim_{n\to\infty} |r - x_n| \leq \lim_{n\to\infty} \rho^n |r - x_0| = 0.$$

□

We have the following result for an upper limit of the error:

Corollary 2 *If i) and ii) are satisfied, we have*

$$|r - x_n| \leq \rho^n \max\{x_0 - a, b - x_0\}$$

and

$$|r - x_n| \leq \frac{\rho^n}{1 - \rho}|x_1 - x_0|$$

for all $n \geq 1$.

Proof: The first statement is given by the fact that $|r - x_0| \le \max\{x_0 - a, b - x_0\}$. To show the other one, let

$$\begin{aligned} r - x_1 = g(r) - g(x_0) &= g(r) - g(x_1) + g(x_1) - g(x_0) \\ &= g'(\xi_1)(r - x_1) + g'(\xi_2)(x_1 - x_0). \end{aligned}$$

Use the triangle inequality and assumption *ii)*,

$$|r - x_1| \le \rho|r - x_1| + \rho|x_1 - x_0|,$$

or

$$|r - x_1| \le \frac{\rho}{1-\rho}|x_1 - x_0|,$$

which leads to

$$|r - x_n| \le \rho^{n-1}|r - x_1| \le \frac{\rho^n}{1-\rho}|x_1 - x_0|.$$

□

Corollary 3 *Assume that $g'(x)$ is continuous close to r.*

- *If $|g'(r)| < 1$ the fixed-point iterations will converge if the starting points are sufficiently good.*
- *If $|g'(r)| > 1$ the fixed-point iterations will not converge.*

We say that a fixed-point scheme converges to r with convergence order k if there is an $M < \infty$ such that

$$|r - x_{n+1}| \le M|r - x_n|^k$$

Theorem 4 *If g is $k \ge 1$ times continuous differentiable at a fixed-point r, and $g'(r) = g''(r) = \ldots = g^{(k-1)}(r) = 0$, while $g^{(k)}(r) \ne 0$, the fixed-point iteration scheme $x_{n+1} = g(x_n)$ converges to r with convergence order k, under the assumption that the starting values are sufficiently close to r.*

Proof: Let $e_n = r - x_n$. Using a Taylor-series expansion of g about r we get

$$
\begin{aligned}
&r - x_{n+1} = g(r) - g(x_n) = g(r) - g(r - e_n) \\
&= g'(r)e_n - \frac{1}{2}g''(r)e_n^2 - \ldots - \frac{(-1)^{k-1}}{(k-1)!}g^{(k-1)}(r)e_n^{k-1} - \frac{(-1)^k}{k!}g^{(k-1)}(r)e_n^k.
\end{aligned}
$$

Since $g^{(k)}(r)$ is continuous about r, there exists an M such that $|g^{(k)}(x)|/k! \leq M$ close to r. We assume that x_n(and in particular x_0) is close enough to r. Then we have

$$|r - x_{n+1}| \leq M|r - x_n|^k$$

Hence, we get convergence of order k if x_0 is sufficiently close to r such that $M|r - x_0|^{k-1} < 1$.