MA2501 Numerical methods

Spring 2010

Problem set 7

Exercise 1

Determine the constants a, b, c and d such that the formula

$$\int_{-1}^{1} f(x)\mathrm{d}x \simeq af(-1) + bf(1) + cf'(-1) + df'(1)$$

is exact for polynomials of a degree as high as possible.

Exercise 2

We have the quadrature formula

$$\int_{-1}^{1} f(x)\mathrm{d}x \simeq f(-\alpha) + f(\alpha). \tag{1}$$

a) What is α if (1) is to be exact for all quadratic polynomials? What about cubic polynomials?

b) Find α such that (1) is exact for all

$$f(x) = a + \sum_{i=1}^{n} b_i x^{2i-1} + cx^{2n}$$

Exercise 3

A quadrature formula is usually given on the form

$$Q(f; a, b) = \sum_{i=0}^{n} w_i f(x_i)$$

where the nodes $x_i \in [a, b]$. The quadrature formula has precision m if

$$\int_a^b p(x)\mathrm{d}x = Q(p; a, b), \qquad \forall p \in \mathbb{P}_m.$$

a) Use the method described on p.230 to construct a quadrature formula $Q(f; -1, 1)$ with nodes $x_0 = -1, x_3 = 1$, and x_1, x_2 equal to the zeros of $L_3'(x)$. Here, $L_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ is the 3. Legendre-polynomial. Determine the precision of the method.

b) Start with the quadrature formula you found in **a)**, and use this to find an approximation to the integral $\int_{t_j}^{t_j+h} f(t)\mathrm{d}t$. Use this to construct a composite quadrature rule based on

$$\int_a^b f(t)\mathrm{d}t = \sum_{j=0}^{m-1} \int_{t_j}^{t_j+h} f(t)\mathrm{d}t, \quad t_j = a + jh, \quad h = \frac{b-a}{m}.$$

Find also an expression for the error in the composite quadrature rule.

(It can be shown that $\int_{-1}^{1} f(x)\mathrm{d}x - Q(f, -1, 1) = -\frac{2}{23625} f^{(6)}(\xi), \quad \xi \in [-1, 1]$)

c) Use the formula from **b)** with $m = 2$ to find an approximation to the integral

$$\int_0^1 \frac{1}{1+t}\mathrm{d}t.$$

Find an upper bound on the error.