MA2501 Numerical methods

Spring 2010

Problem set 8

Exercise 1

Use the Matlab-program `skyt.m` to solve the boundary-value problem

$$x'' + e^x = 0, \qquad x(0) = x(1) = 0$$

Exercise 2

Given the initial-value problem

$$x'' = -2t(x')^2, \qquad x(0) = 1, \quad x'(0) = z.$$

Find $\phi(z) = x(1)$, and use this to solve the boundary-value problem

$$x'' = -2t(x')^2, \qquad x(0) = 1, \quad x(1) = 1 + \pi/4.$$

Hint: Set $\psi = x'$ and solve the initial-value problem analytically.

Exercise 3

(Exam SIF 5040 May 2001)
Let $u(x,t)$ be the solution to the advection-diffusion equation

$$\begin{aligned} &u_t + au_x = bu_{xx} \\ &u(0,t) = 0, \quad u(1,t) = 0, \qquad (t \geq 0) \\ &u(x,t) = g(x), \qquad (0 < x < 1). \end{aligned}$$

Here, a, b are positive constants.

We want to find numerical approximations to the differential equation. Let u_i^n be the numerical approximation to $u(x_i, t_n)$ where $x_i = i \cdot h$, $t_n = n \cdot k$ and h and k are given quantities in the x- and t-direction in a uniform grid. We discretize in the x-direction using central-differences.

a) Use Forward Euler i the time-discretization, and construct a numerical scheme. Show that with the following stability-conditions

$$k \leq h^2/(2b) \text{ og } h \leq 2b/a$$

the scheme satisfies the maximum-principle,

$$\max_i |u_i^{n+1}| \leq \max_i |u_i^n|.$$

b) Use Backward-Euler in the time-discretization and construct an implicit numerical scheme.

Show that this scheme satisfies the maximum-principle if $h < 2b/a$. Explain why this condition is much better than the corresponding conditions for the explicit scheme.

Exercise 4

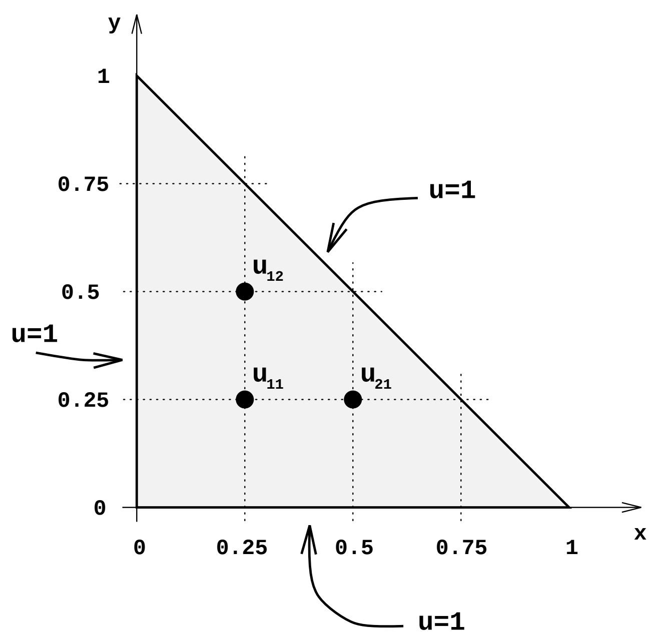
We want to approximate the solution to the partial differential equation (Poisson equation)

$$u_{xx} + u_{yy} = -1$$

in a domain D, where $u(x, y)$ is given on the boundary of D by using a finite difference scheme. The domain D is given by

$$D = \{(x, y) \mid 0 < x < 1, 0 < y < 1 - x\} \; ,$$

and $u(x, y) = 1$ on the boundary. We use step-size $h = 1/4$, and let u_{ij} be the approximation to $u(i \cdot h, j \cdot h)$. See the figure.



a) Find the three equations which determine u_{11}, u_{12} and u_{21} when central differences is used for approximating the derivatives.

The system of equations in **a)** can be written on the form

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{u}$ and $\mathbf{b}$ are vectors.

b) Let $\vec{u} = (u_{11}, u_{21}, u_{12})^T$ (natural ordering of the unknownsin the vector $\mathbf{u}$). Find $\mathbf{A}$ and $\mathbf{b}$. Solve the system and find u_{11}, u_{21} and u_{12}.