Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder
Olivier Verdier

Training Assignment 9

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The purpose of that assignment is to better understand quadrature formulae.

This assignment has 5 tasks.

Exercise 1. Suppose we want to find the zeros of the polynomial:

$$P(x) = -1 + x - x^2 + x^3$$

1.a) Write down the corresponding companion matrix

1.b) Remember that you can create matrices in Python using `array`, so for instance the code

```
M = array([[1.,2.],[3.,4.]])
```

would produce the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Use the function `eigvals` to compute the eigenvalues of the companion matrix of p. Check that these are indeed all the roots of the polynomial p.

Exercise 2. Suppose we choose interpolation points in the interval $[0, 1]$:

$$c_0 = \frac{1}{4} \qquad c_1 = \frac{1}{2} \qquad c_2 = \frac{3}{4}$$

Remember that we obtain a quadrature formula by interpolating a function f at those points, and by integrating exactly the resulting interpolation polynomial. The resulting formula takes the form

$$\int_0^1 f(x)\,\mathrm{d}x \approx I(f) = \sum_{k=1}^{3} w_k f(c_k).$$

2.a) Explain why that quadrature formula integrates exactly the polynomials up to degree 2. Does it depend on the choice of c_0, c_1, c_2?

2.b) Using the preceding fact on the polynomials 1, $x - \frac{1}{2}$, and $(x - \frac{1}{2})^2$ to find directly the weights w_k.

2.c) Write down the Lagrange polynomials ℓ_0, ℓ_1, ℓ_2 for the interpolation points c_k and compute their integrals $w_k = \int_0^1 \ell_k(x)\,\mathrm{d}x$. Do you find the same values of w_k? Explain why.

2.d) Show that the quadrature formula integrates exactly polynomial of degree 3, but not 4. (Hint: use the quadrature formula on the polynomials $(x - \frac{1}{2})^3$ and $(x - \frac{1}{2})^4$)

2.e) Show that the quadrature formula is exact for the polynomial $(x - \frac{1}{2})^5$. Does that mean that it is exact for all the polynomials of degree 5?

2.f) Write the quadrature formula scaled to an arbitrary interval $[a, b]$, in order to approximate $\int_a^b f(x)\,\mathrm{d}x$ (when the interval $[a, b]$ is small)

Exercise 3. We construct a quadrature formula using n points $c_0, \dots, c_{n-1}$. Show that it is impossible to integrate exactly all polynomials of degree $2n$. (Hint: use the quadrature formula on the polynomial M^2, where $M(x) = (x - c_0) \cdots (x - c_{n-1})$.)

Exercise 4. **4.a)** Find two reals c_0, c_1 in $[0, 1]$ such that

$$\int_0^1 (x - c_0)(x - c_1) = 0 \tag{1}$$

4.b) Compute the corresponding weights w_0, w_1

4.c) Show that the corresponding quadrature formula integrates exactly polynomials of degree 2.

4.d) Show that this is in fact true for any choice of c_0 and c_1 as long as (1) is fulfilled. (Hint: use that $x^2 = (x - c_0)(x - c_1) + (c_0 + c_1)x - c_0 c_1$ and the fact that a quadrature formula with two points will always integrate exactly polynomials of degree up to one)

Exercise 5. Suppose that we choose an odd number of interpolation points c_k for $k = 0, \ldots, 2n$. Suppose further that the points are symmetrically placed around $\frac{1}{2}$, i.e., that

$$c_{n-k} + c_{n+k} = 1 \qquad \text{for} \qquad k = 0, \ldots, n.$$

5.a) Show that the weights are symmetric around 1/2, i.e., that

$$w_{n-k} = w_{n+k} \qquad \text{for} \qquad k = 0, \ldots, n.$$

5.b) Show that the quadrature formula integrates exactly polynomials of degree up to $2n + 1$.