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MA2501 Numeriske Metoder
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Training Assignment 11

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The purpose of those exercises is to become familiar with the discrete Fourier transform and its corresponding algorithm, the fast Fourier transform.

Some useful formulae and definitions:

$$\omega_N := \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}}$$

The Fourier matrix of size N is defined as

$$F_N = [\omega_N^{ij}]_{i,j=0,\ldots,N-1}$$

The discrete Fourier transform of $z = (z_0, \ldots, z_{N-1})$ is defined by

$$y_k = \sum_{j=0}^{N-1} z_k \omega_N^{kj},$$

so in matrix vector notation, this is simply

$$y = F_N z.$$

This assignment has 5 tasks.

Exercise 1. 1.a) Compute ω_2, ω_4 and ω_8.

$$\omega_2 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{2}} = \mathrm{e}^{\mathrm{i}\pi} = -1$$
$$\omega_4 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{4}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$$
$$\omega_8 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{8}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$$

1.b) Compute ω_8^8, ω_8^9, and more generally, ω_N^{N+1}.

1.c) Compute ω_{2n}^n

Exercise 2. 2.a) Write down the matrices F_2 and F_4.

2.b) Compute the discrete Fourier transform of

$$z = (1, -1, 1, -1).$$

What do you notice? What is the explanation?

2.c) Create new two by two submatrices from F_4 by following the following prescriptions:

A_{00}**:** First two rows and even columns

A_{10}**:** First two rows and odd columns

A_{10}**:** Last two rows and even columns

A_{11}**:** Last two rows and odd columns

Express those submatrices from F_2 and from the matrix Ω_2 defined as

$$\Omega_2 := \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{bmatrix}.$$

Exercise 3. Suppose we have a periodic function $f(x)$ which we want to approximate as a sum

$$f(\theta) = \sum_{k=0}^{N-1} a_k \cos\left(\frac{2\pi}{N} k\theta\right)$$

How would you use the discrete Fourier transform for that?

Exercise 4. Recall the fast Fourier transform formula, if

$$y = F_{2n} z,$$

then

$$y_j = \sum_{k=0}^{n-1} (z'')_k \omega_n^{kj} + \omega_{2n}^j \sum_{k=0}^{n-1} (z')_k \omega_n^{kj} \tag{1}$$

where z'' is composed of the even components of z, and z' is composed of the odd components of z, that is

$$(z'')_j := z_{2j}, \qquad (z')_j := z_{2j+1}.$$

4.a) Write z' and z'' for a vector $z = (7, 6, 5, 4, 3, 2, 1, 0)$.

4.b) Compute the formula (1) for $n = 2$. Make sure to group together the values $z''_0 + z''_1$, $z''_0 - z''_1$ and $z'_0 + z'_1$, $z'_0 - z'_1$.

4.c) Show that for $j = 0, \dots, n-1$,

$$y_{j+n} = \sum_{k=0}^{2n-1} (z'')_k \omega_n^{kj} - \omega_{2n}^{j} \sum_{k=0}^{2n-1} (z')_k \omega_n^{kj}.$$

Exercise 5. Use the Fast Fourier Transform to compute

$$F_4 z$$

where

$$z = (0, 1, 1, 0)$$

and check that the result is correct by computing the matrix multiplication $F_4 z$ directly.