Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder
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Exam in Numerical Methods (MA2501)

2011-05-25, 09:00 – 13:00

Grading

Maximize your points by answering to as many subquestion as you can. Beware that some subquestions are easier than others. You may answer to the problems and subquestions in any order you like.

All the subquestions weigh the same in the final grade.

Allowed aids

- Cheney & Kincaid, *Numerical Mathematics and Computing*, 5. or 6. edition
- Rottmann, *Mathematical Formulae*
- Approved calculator

Problem 1. Consider the function

$$f(x) = |x|$$

We want to interpolate this function at the points

$$-2, -\frac{3}{2}, \frac{1}{2}, 1$$

1.a) Express the interpolation polynomial using the Lagrange polynomials.

Write the Lagrange polynomials, for instance for p_1:

$$p_1(x) = \frac{(x+3/2)(x-1/2)(x-1)}{(-2+3/2)(-2-1/2)(-2-1)} = ..$$

The interpolation polynomial is then given by:

$$p(x) = \sum_i f(x_i)p_i(x) = 2p_1(x) + \frac{3}{2}p_2(x) + \frac{1}{2}p_3(x) + p_4(x)$$

1.b) Use Newton's divided difference method to find the interpolation polynomial.

The table of the divided differences is given by:

$$\begin{array}{ccccc}
-2 & 2 & & & \\
 & & -1 & & \\
-\frac{3}{2} & \frac{3}{2} & & \frac{1}{5} & \\
 & & -\frac{1}{2} & & \frac{2}{15} \\
\frac{1}{2} & \frac{1}{2} & & \frac{3}{5} & \\
 & & 1 & & \\
1 & 1 & & &
\end{array}$$

We thus read out that the interpolation polynomial is

$$p(x) = 2 - (x+2) + \frac{1}{5}(x+2)(x+3/2) + \frac{2}{15}(x+2)(x+3/2)(x-1/2)$$

Problem 2. Consider the differential equation

$$u''(t) = f(u(t))$$

for an arbitrary function f defined on $\mathbf{R}$.

2.a) Write one step of the explicit Euler method (you must first convert the differential equation into a system of first order differential equations)

We have to rewrite the differential equation into a *system* of differential equations, using an auxiliary variable v:

$$u' = v$$
$$v' = f(u(t))$$

We now have a first order differential equations that we can discretize using the explicit Euler method:

$$u_1 = u_0 + hv_0$$
$$v_1 = v_0 + hf(u_0)$$

2.b) Write, in the language of your choice, a function that implements one step of the Euler method.

In Python, for instance, one possibility would be

```
def euler(f,x,h):
  return x + h*f(x)
```

Problem 3. We want to solve the following equation in u_1 (u_1 is the unknown, whereas u_0 and h are known):

$$u_1 = u_0 + h\cos(u_1)$$

3.a) Run one iteration of Newton's method to find an approximation of u_1.

Let us define the function

$$F(u) = u - u_0 - h\cos(u)$$

We are looking for a solution of

$$F(u) = 0.$$

Observe that

$$F'(u) = 1 + h\sin(u)$$

One step of Newton's method is therefore given by:

$$\widetilde{u} := u - F(u)/F'(u) = u - \frac{u - u_0 - h\cos(u)}{1 + h\sin(u)}$$

3.b) Assume for instance that $u_0 = 1.$ and $h = 0.1$. Choose an initial guess for Newton's method and compute the result after one step of Newton's method using the formula in question **3.a**.

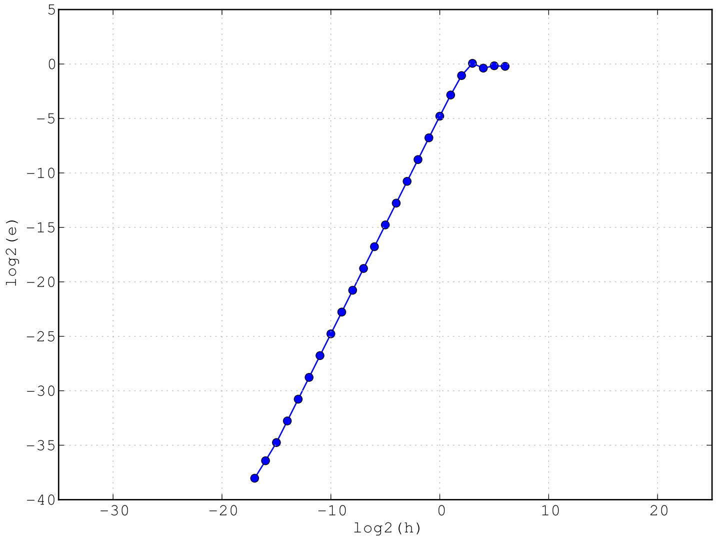
$$\widetilde{u} = u_0 - \frac{-h\cos(u_0)}{1 + h\sin(u_0)} \approx 1.05$$

Problem 4. We are studying the accuracy of a given formula $\varphi(h)$, which approximates the exact value x. In order to estimate how fast the formula $\varphi(h)$ converges to x with respect to h, we plot $\log(|\varphi(h) - x|)$ with respect to $\log(|h|)$. The resulting plot is in the figure below.

Deduce from the figure that for h sufficiently small we have

$$|\varphi(h) - x| \approx Ch^n$$

(where C is a constant), and provide a value for n.



The graph is a line of slope 2, so we may write

$$\log_2(|\varphi(h) - x|) = 2\log_2(h) + K,$$

where K is a constant.

By taking the exponential of the previous equation we obtain

$$|\varphi(h) - x| = \mathrm{e}^K h^2,$$

from which we obtain the desired result, with $C = \mathrm{e}^K$. In particular, we obtain

$$n = 2.$$

Problem 5. On the interval $[0,1]$ we choose the interpolation points

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}$$

One obtains a local integration formula, i.e., an approximation of $\int_0^1 f(x)\,\mathrm{d}x$, by interpolating f at the interpolation points, and by integrating exactly the resulting interpolation polynomial. The resulting formula takes the form

$$I(f) = \sum_{i=1}^{3} w_i f(x_i)$$

5.a) Find out the weights w_i of the formula above.

The formula has to integrate exactly the polynomials 1, $x - 1/2$ and $(x-1/2)^2$. From the second assertion we obtain $w_1 - w_3 = 0$. The last one gives

$$\int_0^1 (x-1/2)^2\,\mathrm{d}x = 1/24 = 1/16(w_1 + w_3) = w_1/8$$

This gives

$$w_1 = \frac{2}{3}$$

We know that $w_3 = w_1$, so

$$w_3 = \frac{2}{3}$$

Finally, from the assertion that constants are integrated exactly, we obtain $w_1 + w_2 + w_3 = 1$, so $w_2 = -\frac{1}{3}$.

We have thus obtained:

$$w_1 = \frac{2}{3} \qquad w_2 = -\frac{1}{3} \qquad w_3 = \frac{2}{3}$$

5.b) Show that this formula integrates exactly polynomials of degree lower than or equal to 3, but not exactly all polynomials of degree 4.

It is straightforward to check that the formula is exact for the polynomial $(x-1/2)^3$, and since it is exact for all polynomials for degree lower than or equal to two, it is exact for all the polynomials of degree lower than or equal to three.

For the polynomial $(x-1/2)^4$ we have

$$\int_0^1 (x-1/2)^4 = \frac{1}{5}\frac{1}{2^4} \neq 2\frac{2}{3}\left(\frac{1}{4}\right)^4 = \frac{1}{3}\frac{1}{2^6}$$

so the formula does not integrate exactly polynomials of degree 4.

Problem 6. We consider the problem

$$-u'' + u = f$$
$$u(0) = 0$$
$$u(1) = 0$$

where u is the unknown and f is a given function.

We choose N linearly independent functions φ_i defined on the interval $[0, 1]$.

The matrix A is defined by its entries

$$a_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x) + \int_0^1 \varphi_i(x)\varphi_j(x)\,\mathrm{d}x, \qquad 1 \leq i, j \leq N.$$

The matrix A may be used to solve the differential equation above: if $Ax = b$ and the vector x has coordinates $x = (x_1, \ldots, x_N)$, then $u = \sum_{i=1}^N x_i\varphi_i)$ is an approximation to the solution of the original problem.

Show that the matrix A is invertible.

We are going to show that the square matrix A has no null vectors, i.e., $Ax = 0 \implies x = 0$. Since it is a square matrix, it will imply invertibility.

Let us assume that

$$Ax = 0$$

for a vector of coordinates $(x_1, \ldots, x_N)$.

We can define the corresponding function u by

$$u(x) = \sum_{i=1}^N x_i\varphi_i(x)$$

If $Ax = 0$ then

$$\alpha_i = \sum_j a_{ij}x_j = 0 \qquad \forall i$$

In particular:

$$\sum_i \alpha_i x_i = 0$$

Note that

$$\alpha_i = \sum_j \Big(\int \varphi_i'\varphi_j' + \int \varphi_i\varphi_j\Big)x_j = \int \varphi_i' u' + \int \varphi_i u$$

We thus have

$$0 = \sum_i \alpha_i x_i = \sum_i \Big(\int \varphi_i' u' + \int \varphi_i u\Big)x_i = \int u'^2 + \int u^2$$

Since both terms are positive, this means that $u = 0$. This means:

$$u = \sum_i x_i \varphi_i = 0$$

and since the functions φ_i are linearly independent, this implies that

$$x_i = 0 \qquad \forall i$$

which means that the vector x is zero, and thus, that A is invertible.