Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder

Olivier Verdier (contact: 48 95 20 66)

Exam in Numerical Methods (MA2501)

2012-06-08, 15:00 – 19:00

Grading

- Maximize your points by answering to as many subquestion as you can.
- Some subquestions are easier than others.
- You may answer to the problems and subquestions in any order you like.

All the ten subquestions weigh the same in the final grade.

If you give at least four correct answers, you will pass the exam.

Allowed aids

- Cheney & Kincaid, *Numerical Mathematics and Computing*, 5. or 6. edition
- Rottmann, *Mathematical Formulae*
- Approved calculator

Problem 1. Consider the interpolation points

x	1	2	3	4
y	1	5	15	7

1.a) Compute the Lagrange polynomial for the interpolation point $x = 4$.

Expected answer (motivated): One polynomial, in whichever form you like.

The definition of a Lagrange polynomial should be perfectly clear. It is *not* the interpolation polynomial. It is the lowest degree polynomial which is one at the point $x = 4$ and zero at the other interpolation points. We get

$$\boxed{\ell(x) = \frac{(x-1)(x-2)(x-3)}{6}}$$

1.b) Use divided differences to compute the interpolating polynomial (of minimum degree).

Expected answer (motivated): A polynomial in whichever form you like and a divided difference table.

1	1			
		4		
2	5		3	
		10		-4
3	15		-9	
		-8		
4	7			

The polynomial is therefore

$$\boxed{P(x) = 1 = 4(x-1) + 3(x-1)(x-2) - 4(x-1)(x-2)(x-3)}$$

Problem 2. Consider the differential equation

$$u'(t) = -u(t) + t + \frac{1}{2}$$

with initial condition $u(0) = 1$. Use the explicit Euler method (Euler's method in the book) in order to compute an approximation of $u(0.2)$, using a time step $h = 0.1$.

Expected answer (motivated): One number; three significant digits are enough.

We have

$$u_1 = 1 + 0.1(-1 + \frac{1}{2}) = 0.95$$

and

$$u_2 = 0.95 + 0.1(-(0.95) + 0.1 + \frac{1}{2}) \approx 0.92$$

Problem 3. Consider the nodes

$$c_1 = \frac{1}{6}, \qquad c_2 = \frac{1}{2}, \qquad c_3 = \frac{5}{6},$$

and the corresponding quadrature formula

$$Q(f) = w_1 f(c_1) + w_2 f(c_2) + w_3 f(c_3)$$

which approximates the integral $\int_0^1 f(x)\,\mathrm{d}x$.

3.a) Determine the weights w_1, w_2, w_3 so that the formula is exact for polynomials of degree up to two, that is

$$Q(P) = \int_0^1 P(x)\,\mathrm{d}x$$

if P is a polynomial of degree up to two.

Expected answer (motivated): Three numbers w_1, w_2, w_3.

First, with the polynomial 1 we obtain $w_1 + w_2 + w_3 = 1$. Then with $P = x - 1/2$ we obtain $w_1/9 - w_3/9 = 0$, that is $w_1 = w_3$. Next, with $P = (x - 1/2)^2$ we get $w_1/9 + w_3/9 = 1/12$, that is $w_1/3 + w_3/3 = 1/4$. By using $w_1 = w_3$ we get that $w_1 = w_3 = 3/8$. The final weight is thus $w_2 = 1/4$.

To summarise:

$$\boxed{w_1 = \frac{3}{8} \qquad w_2 = \frac{1}{4} \qquad w_3 = \frac{3}{8}}$$

3.b) Compute the error $E_k \coloneqq \left|Q(x^k) - \int_0^1 x^k\,\mathrm{d}x\right|$ for the lowest integer k such that E_k is not zero.

Expected answer (motivated): An integer k and a number E_k.

One can check that $Q((x-1/2)^3) = 0 = \int_0^1 (x-1/2)^3$, so the formula is still exact for $k = 3$. We obtain by direct calculation

$$\boxed{E_4 = \left|\frac{3}{8}\frac{1}{6^4} + \frac{1}{4}\frac{1}{2^4} + \frac{3}{8}\frac{5^4}{6^4} - \frac{1}{5}\right| = \frac{7}{2160} \approx 3.24 \times 10^{-3}}$$

and

$$\boxed{k = 4}$$

Problem 4. Given a fixed real number ω, consider the function $F_\omega \colon \mathbf{R}^2 \to \mathbf{R}^2$ defined as

$$F_\omega(x, y) \coloneqq (x^2 - y + \omega, y^2 - x + \omega).$$

4.a) Write down Newton's algorithm to find the solution of the problem

$$F_\omega(x, y) = 0.$$

Write the algorithm in the form

$$M(x_n, y_n)\Delta X = v(x_n, y_n)$$

where the vector $v(x_n, y_n)$ and the matrix $M(x_n, y_n)$ depend only on x_n, y_n (and ω), and ΔX is the vector defined as $\Delta X \coloneqq (x_{n+1} - x_n, y_{n+1} - y_n)$.

Expected answer (motivated): One 2×2 matrix M and one vector v, both depending on x_n, y_n and ω.

The Jacobian of F_ω is

$$F'_\omega(x, y) = \begin{bmatrix} 2x & -1 \\ -1 & 2y \end{bmatrix}$$

Newton's algorithm is

$$F'(x_n)\Delta x = -F(x_n)$$

So the final system is

$$\begin{bmatrix} 2x_n & -1 \\ -1 & 2y_n \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} x_n^2 - y_n + \omega \\ y_n^2 - x_n + \omega \end{bmatrix}$$

We deduce that

$$\boxed{M = \begin{bmatrix} 2x_n & -1 \\ -1 & 2y_n \end{bmatrix}}$$

$$\boxed{v = -(x_n^2 - y_n + \omega, y_n^2 - x_n + \omega)}$$

4.b) Suppose that $\omega = 1$ and choose the initial value $x_0 = 0$, $y_0 = 0$. Run one iteration of Newton's algorithm.

Expected answer (motivated): Two numbers x_1, y_1.

The system to solve is

$$-\Delta y = -\omega$$
$$-\Delta x = -\omega$$

which gives $x_1 = \omega$, $y_1 = \omega$. For $\omega = 1$ this gives $x_1 = 1$, $y_1 = 1$. So

$$\boxed{x_1 = 1 \qquad y_1 = 1}$$

Problem 5. For a given number $\varepsilon \neq 1$, consider the matrix A defined by

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1+\varepsilon & 3 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

5.a) Compute the LU decomposition of the matrix A (without any pivoting).

Expected answer (motivated): Two 3×3 matrices L and U which depend on ε.

Note that for this question we implicitly assume that $\varepsilon \neq 0$. The first multipliers are $\ell_{21} = 1$, $\ell_{31} = -0.5$. The new submatrix is

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$$

At the next step, the multiplier is $\ell_{23} = 1/\varepsilon$, and the final submatrix is just $1 - 1/\varepsilon$. We deduce that

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -0.5 & 1/\varepsilon & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & \varepsilon & 1 \\ 0 & 0 & 1 - 1/\varepsilon \end{bmatrix}}$$

5.b) Use the LU decomposition in order to find a solution x to the problem $Ax = b$ for the vector

$$b = (1, 2, 1).$$

Expected answer (motivated): One vector x depending on ε; you must explain how you use the LU decomposition.

The general procedure is to decompose $Ax = b$ into first $Ly = b$ and then $Ux = y$, thus

$$\boxed{Ly = b \qquad Ux = y}$$

The solution of $Ly = b$ is, by backward substitution:

$$y = (1, 1, 3/2 - 1/\varepsilon)$$

Now the solution of $Ux = y$ by subsequent backward substitution is simply

$$\boxed{x = \left(\frac{-3/2 + 2\varepsilon}{1-\varepsilon}, \frac{1/2}{1-\varepsilon}, \frac{1 - 3\varepsilon/2}{1-\varepsilon}\right)}$$

Note that even though the LU decomposition previously obtained is undefined for $\varepsilon = 0$, the *result* x is defined whether or not $\varepsilon = 0$. However, when $\varepsilon = 1$, the matrix is not invertible, and the problem has no solution; accordingly, the formula above is not defined for $\varepsilon = 1$.

5.c) Repeat the last question *but* taking into account that ε is a number which is so small that $1 + \varepsilon = 1$ on the floating point system of the computer.

Expected answer (motivated): One vector x.

Let us redo the last question from a computer's point of view, taking into account that $1 + \varepsilon = 1$ in the floating point system.

a) The first equation gives directly: $y_1 = 1$

b) The second equation is $y_1 + y_2 = 2$ and thus yields $y_2 = 1$

c) The third equation, $y_1/2 + y_2/\varepsilon + y_3 = 1$ so $1/2 + 1/\varepsilon + y_3 = 1$, so $y_3 = 3/2 - 1/\varepsilon$. Now, using that $1 + \varepsilon = 1$ we get $3/2 - 1/\varepsilon \approx -1/\varepsilon$ so $y_3 = -1/\varepsilon$.

So we have obtained

$$\boxed{y = (1, 1, -1/\varepsilon)}$$

Let us now turn to $Ux = y$.

a) First, $-x_3/\varepsilon = -1/\varepsilon$ yields $x_3 = 1$

b) Then, $\varepsilon x_2 + x_3 = y_2$ becomes $\varepsilon x_2 + 1 = 1$, so $x_2 = 0$

c) Finally, $x_1 + x_2 + 2x_3 = 1$ gives $x_1 = -1$.

Note that the only place where we used that $1+\varepsilon = 1$ is for the calculation of y_3, and that error gets amplified in the rest of the computations. We have obtained:

$$\boxed{x_{\text{computed}} = (-1, 0, 1)}$$

Compare the true solution obtained in **5.b** with $\varepsilon = 0$, and the one you just computed. How big is the norm of the error expressed as a percentage of the norm of the true solution?

Expected answer (motivated): A percentage.

With exact calculation one obtains

$$\boxed{x_{\text{exact}} = (-3/2, 1/2, 1)}$$

whereas the approximate solution is

$$\boxed{x_{\text{computed}} = (-1, 0, 1)}$$

The error vector is

$$\Delta x = (1/2, -1/2, 0)$$

Its norm of the error is therefore $1/\sqrt{2}$. The norm of the exact solution is $\sqrt{14}/2$, so the relative error is

$$\frac{\sqrt{2}}{\sqrt{14}} \approx 38 \times 10^{-2}$$

The relative error is therefore

$$\boxed{E = 38\%}$$

This "error" is *huge*. The culprit is the algorithm, not the matrix. Using pivoting would produce a very good approximation.