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MA2501 Numeriske Metoder
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Training Assignment 1

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This assignment has 6 tasks.

Exercise 1. We are looking for a solution of the equation $x^3 + x - 2 = 0$.

1.a) Write down one step of Newton's method.

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 2}{3x_n + 1}$$

1.b) Compute the result after two steps, starting from the initial guess $x_0 = 0$.

1.c) Write down one step of Newton's method as an expression computable in a computer, as was done in the lecture.

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x = x - (x**3 + x - 2)/(3*x +1)
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Exercise 2. How many steps are necessary for Newton's method to converge when computing the root of $f(x) = ax + b$? Does it depend on the starting point?

$$x_{n+1} = x_n - \frac{ax_n + b}{a} = -\frac{b}{a}$$

so after one step, no matter what the starting point is, Newton's method yields $-\frac{b}{a}$, which is the solution.

Exercise 3. Write down Newton's method for the function $f(x) = \frac{1}{x}$. What happens? Calculate the result after 50 iterations, assuming that the starting value is 1.

$$x_{n+1} = x_n + \frac{x_n^2}{x_n} = 2x_n$$

so

$$x_n = 2^n x_0$$

For $x_0 = 1.$ we find $x_{50} = 2^{50}$.

Exercise 4. The sequence u_n is defined implicitly, for a given function f and a small, positive real number h, as

$$\frac{u_{n+1} - u_n}{h} = f(u_{n+1}).$$

4.a) If you also fix u_n, find a function of which u_{n+1} is the root.

$$F(x) = x - u_n - hf(x)$$

4.b) Write down one step of Newton's method, that should converge to that root.

$$x_{m+1} = x_m - \frac{F(x_m)}{F'(x_m)} = x_m - \frac{x_m - u_n - hf(x_m)}{1 - hf'(x_m)}$$

4.c) What initial condition would you choose to start the iteration?

If h is small, then u_n is almost a solution to the equation $u_{n+1} = u_n + hf(u_{n+1})$, so u_n seems to be a good candidate.

Exercise 5. A continuous functions which changes its sign in an interval $[a, b]$, i.e. $f(a)f(b) < 0$, has at least one zero, or *root*, in this interval. In other words, there exists at least one point $r \in [a, b]$ such that

$$f(r) = 0.$$

Such a root r can be found by the *bisection method*, which we describe now.

This method starts from the given interval $[a, b]$. Then it investigates the sign changes in the subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. If the sign changes in the first subinterval b is redefined to be

$$b := \frac{a+b}{2}$$

otherwise, a is redefined in the same manner to

$$a := \frac{a+b}{2},$$

and the process is repeated until the quantity $|b-a|$ is less than a given tolerance.

Show that the bisection method converges, if the assumption are fulfilled.

1. Notice first that the algorithm produces new intervals, no matter the function. Let us denote these intervals by $[a_n, b_n]$.

2. Notice also that $a_n \leq b_0$ for all n and that a_n is an increasing sequence. It is a standard result that a_n converges to a number c.

3. One shows by induction that $|b_n - a_n| = \frac{|b_0 - a_0|}{2^n}$. As a result, b_n converges as well, to the same value c.

4. Finally, suppose for instance that $f(a_0) > 0$ and that $f(b_0) < 0$. The algorithm is such that $f(a_n) \geq 0$ and $f(b_n) \leq 0$, so we obtain $f(c) \geq 0$ and $f(c) \leq 0$ by continuity of f, so $f(c) = 0$.

Exercise 6. Assuming that Newton's method and bisection method converge, one can show the following estimates of their respective errors, *provided that the initial error* e_0 *is sufficiently small*:

$$\begin{aligned} |e^N_{n+1}| &\leq C|e^N_n|^2 && \text{Newton} \\ |e^B_{n+1}| &\leq C' \frac{|e^B_n|}{2} && \text{Bisection} \end{aligned}$$

Which method, in general, will be fastest to converge?