

MA2501 Numeriske Metoder
Olivier Verdier

Training Assignment 4

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This assignment has 4 tasks.

Exercise 1. Consider the interpolation points

$$\begin{array}{c|ccc} x & 0 & 2 & 3 \\ \hline y & 1 & 1 & \frac{5}{2} \end{array}$$

- 1.a)** Compute the interpolating polynomial using a linear combination of Lagrange polynomials. Check that this polynomial indeed interpolates the points.

$$P(x) = \ell_0(x) + \ell_1(x) + \frac{5}{2}\ell_2(x)$$

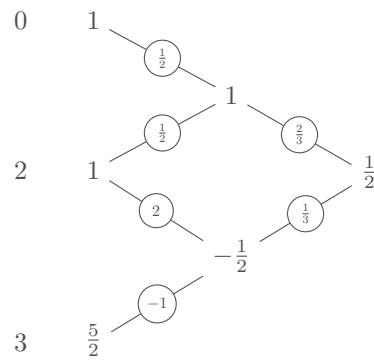
So

$$P(x) = \frac{x-2}{-2} \frac{x-3}{-3} + \frac{x}{2-1} \frac{x-3}{2-3} + \frac{5}{2} \frac{x-0}{3-0} \frac{x-2}{3-2}$$

A slightly cumbersome calculation shows that

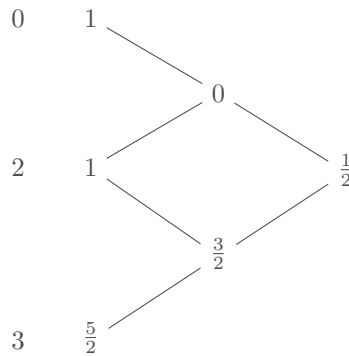
$$P(x) = \frac{1}{2}((x-1)^2 + 1)$$

- 1.b)** Compute the value of $P(1)$ using the Neville algorithm. Compare with the value you get from the polynomial you previously computed.



We conclude that $P(1) = \frac{1}{2}$, which is confirmed by the previous calculation.

1.c) Compute the interpolating polynomial using Newton's divided differences. Compare with the solution you already know.



We deduce that the Newton polynomial is given by

$$P(x) = 1 + \frac{1}{2}x(x - 2)$$

Exercise 2. Given the interpolation points x_0, \dots, x_n , one defines the corresponding Lagrange polynomials ℓ_0, \dots, ℓ_n as the unique polynomial such that

$$\ell_k(x_j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

2.a) Recall the expression of the Lagrange polynomial ℓ_k

2.b) Show (without calculation) that

$$\ell_0 + \ell_1 + \cdots + \ell_n = 1$$

The polynomial

$$P(x) = \ell_0(x) + \ell_1(x) + \cdots + \ell_n(x)$$

is such that

$$P(x_k) = 1$$

so it interpolates the $n + 1$ points $(x_k, 1)$. Now P is a sum of polynomials of degree n , so it is also of degree n . There is a unique polynomial of degree n interpolating $n + 1$ points, and the constant polynomial 1 is such a polynomial, so we conclude that $P = 1$.

2.c) Show (without calculation) that

$$x_0\ell_0(x) + x_1\ell_1(x) + \cdots + x_n\ell_n(x) = x$$

This is exactly the same reasoning as before. The polynomial

$$P(x) = x_0\ell_0(x) + x_1\ell_1(x) + \cdots + x_n\ell_n(x) - x$$

is such that $P(x_k) = 0$. Now P has degree n and is zero on $n + 1$ points, so it must be zero.

Exercise 3. You should follow this instructions on the Python installation page of the course. In order to plot a function in the interval $[0, 1]$ with 500 points, for example the function $x \mapsto x^2$, you may use

```
xs = linspace(0., 1., 500)
ys = xs**2
plot(xs, ys)
```

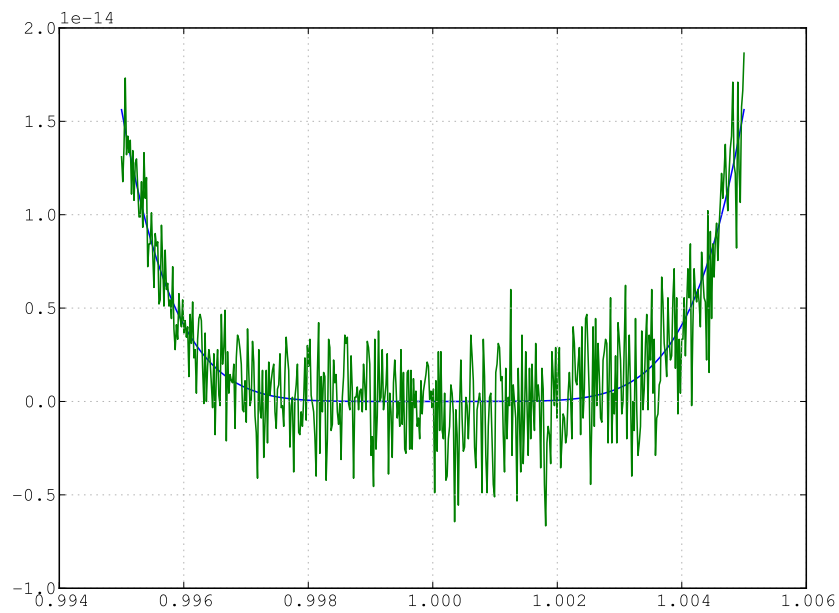
Try that first.

3.a) Plot between 0.995 and 1.005 the function $x \rightarrow (x - 1)^6$ calculated by

```
ys = (xs - 1)**6
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3.b) Do the same with the *same* function but expanded as $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

What do you observe? How do you explain that?



Exercise 4. Consider the functions $T_n(x)$, defined on $[-1, 1]$ by

$$T_n(x) = \cos(n \cdot \arccos(x)), \quad \text{for } n = 0, 1, 2, \dots$$

Show that they satisfy the recursion formula

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

Conclude that the function T_n are in fact *polynomials*. (Hint: use the auxiliary variable $x = \cos(\theta)$)