Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder
Olivier Verdier

Training Assignment 7

2012-03-01

This assignment has 7 tasks.

Exercise 1. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 5 & -1 \\ 6 & 3 & 7 \end{bmatrix}.$$

1.a) Compute the matrices $L^{(1)}$ and $L^{(2)}$ as in the lecture for the matrix A, that is, so that the matrices $A^{(1)} := L^{(1)}A$ and $A^{(2)} := L^{(2)}A^{(1)}$ are the intermediate matrices obtained in the Gauss elimination.

The first matrix is

$$L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

This gives

$$L^{(1)}A = \begin{bmatrix} 3 & 4 & 3 \\ 0 & \frac{11}{3} & -2 \\ 0 & -5 & 1 \end{bmatrix}$$

So the last matrix is

$$L^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{15}{11} & 1 \end{bmatrix}$$

1.b) Use the previous result to give the decomposition

$$A = LU$$

where L is lower triangular with ones on the diagonal, and U is upper triangular.

Check that the multiplication of L and U actually gives A!

Following the lecture, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 2 & -\frac{15}{11} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

1.c) Use the LU decomposition above to solve a specific problem, for instance

$$Ax = b$$

where $b = (10, 7, 15)$.

We solve using an intermediate variable y, so first

$$Ly = b$$

by backward substitution, and then

$$Ux = y$$

Exercise 2. What would you choose between an algorithm which costs $30n^2$ multiplications, and an algorithm which costs n^3 multiplications, for a problem of size n?

If n is big enough, the algorithm which costs $30n^2$ will be vastly less expensive than the other one.

Exercise 3. Consider the matrix

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \ell_{21} & 1 & 0 & & \\ \ell_{31} & 0 & 1 & \ddots & \\ \vdots & & & \ddots & 0 \\ \ell_{n1} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

What is the inverse of the matrix $\mathcal{L}$? (Hint: perform a Gauss elimination on L)

We would obtain for $L^{(1)}$ the same matrix as $\mathcal{L}$ but with all the ℓ coefficients with opposite signs. The next matrices will be $L^{(n)} = I$. We conclude that $I = L^{(1)}\mathcal{L}$, so the inverse of $\mathcal{L}$ is simply $L^{(1)}$.

Exercise 4. Consider the matrix

$$\mathcal{L} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 1 & d_2 & 0 & \cdots & 0 \\ 0 & 1 & d_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & d_n \end{bmatrix}$$

Compute the LU decomposition of the matrix $\mathcal{L}$.

Exercise 5. Compute the condition number of the matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

for arbitrary numbers α and β, first when $\alpha < \beta$, then when $\alpha > \beta$, then when $\alpha = \beta$. For which values of α and β is the condition number as low as possible?

Exercise 6. **6.a)** Can you compute the LU decomposition when the matrix A is not invertible?

6.b) How would you compute the determinant of the matrix A, if you know the LU decomposition?

Exercise 7. The determinant of a matrix is zero when the matrix is not invertible. Why is it preferable to use the condition number rather than the determinant to determine how unwieldy a matrix is?