Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder

Olivier Verdier

Training Assignment 8

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This assignment has 4 tasks.

Exercise 1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 & 3 \\ 1 & -2 & 2 & 4 \\ -2 & 4 & -2 & 2 \\ -1 & 6 & -9 & 5 \end{bmatrix}.$$

1.a) Change the order of the rows to obtain a new matrix A' such that

$$A' = LU$$

with L lower triangular with ones on the diagonal, U upper triangular, and the coefficients in the lower triangular matrix L are lower than or equal to one.

The procedure to achieve that is to perform a Gauss elimination with partial pivoting. Which rows are actually pivoted will give the final permutation. The first row is number three, so it is placed first. After one round of Gauss elimination, we obtain the submatrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 5 \\ 4 & -8 & 4 \end{bmatrix}$$

The multipliers are $(0., -0.5, 0.5)$. The next biggest row is now the last one, so we place it first, and obtain for the next round the mulltipliers $(1/4, 0)$, and the matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$$

Both pivot have the same value, so we may keep that order and obtain for the final round of the elimination the multiplier 1 and the final matrix with one element 3.

This means that by reordering the rows in the order $(3, 4, 1, 2)$, we obtain a LU decomposition as desired. For example, the first multipliers were found to be $(0, -1/2, 1/2)$, but after permutation that would be $(1/2, 0, -1/2)$ because the row number four is now placed first. Reading the multipliers in that way, we obtain that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 1/4 & 1 & 0 \\ -1/2 & 0 & 1 & 1 \end{bmatrix}$$

The matrix U is similarly

$$\begin{bmatrix} -2 & 4 & -2 & 2 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and the matrix A' is

$$A' = \begin{bmatrix} -2 & 4 & -2 & 2 \\ -1 & 6 & -9 & 5 \\ 0 & 1 & -1 & 3 \\ 1 & -2 & 2 & 4 \end{bmatrix}$$

1.b) Find a permutation matrix P such that

$$A = PA',$$

so that we obtain the final decomposition $A = PLU$.

The permutation corresponding to the row permutation $(3, 4, 1, 2)$ is

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

One can check that $A = PA'$.

1.c) Use the `lu` function in Python or Matlab on the matrix A. Do you obtain the same result?

1.d) Use your decomposition PLU to solve the problem

$$Ax = b$$

where

$$b = (1, -1, 6, -1)$$

Notice that solving $Ax = b$ is equivalent to $PA'x = b$, and if we denote $b' = P^{-1}b$, we just have to solve

$$A'x' = b'.$$

We apply the inverse permutation to obtain $b' = (6, -1, 1, -1)$. Now we solve $LUx' = b'$, first by solving $Ly = b'$, we obtain

$$y = (6, -4, 2, 0)$$

Now we solve $Ux = y$ and obtain

$$x = (1, 3, 2, 0)$$

Exercise 2. Consider the problem

$$Ax = b$$

where

$$A = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 9 & -2 \\ -2 & -1 & 8 \end{bmatrix} \qquad b = \begin{pmatrix} 4 & 6 & 5 \end{pmatrix}.$$

Compute the first two steps of the Jacobi and Gauss-Seidel methods for that problem (either by hand or with a computer), with the initial value $x_0 = (0, 0, 0)$.

Exercise 3. Suppose that for a matrix T we have

$$\max_{\|z\|=1} \|Tz\| < 1.$$

Show that the fixed point iteration method $x_{n+1} = Tx_n + c$ is convergent (for any fixed vector c and any initial value x_0).

Let us denote the fixed point (i.e., the solution of the problem) by x. The error at the step n is then $E_n := x_n - x$. We obtain

$$E_{n+1} = x_{n+1} - x = Tx_n + c - (Tx + c) = T(x_n - x) = TE_n$$

Now we have

$$\|T(E_n)\| = \|E_n\| \|T(z_n)\|$$

with $z_n = \frac{E_n}{\|E_n\|}$. Notice that $\|z_n\| = 1$, so $\|T(z_n)\| \le \max_{\|z\|=1} \|T(z)\| < 1$. So, if we define $C := \max_{\|z\|=1} \|T(z)\|$, we have

$$\|E_{n+1}\| \le C\|E_n\|$$

so by induction we obtain $\|E_n\| \le C^n\|E_0\|$. As a result, since $C < 1$, we get that E_n goes to zero, which means that x_n converges towards x.

Exercise 4. We want to solve numerically the problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = 1$$

on the interval $[0, 1]$. After discretization with N points in the interval $[a, b]$, the problem is reduced to the linear problem

$$A_N x = b_N.$$

The matrix A_N has size $N \times N$ and has $2N^2 + 1$ on the diagonal, and $-N^2$ on the upper and lower diagonal:

$$A_N = \begin{bmatrix} 2N^2+1 & -N^2 & & \\ -N^2 & \ddots & \ddots & \\ & \ddots & \ddots & -N^2 \\ & & -N^2 & 2N^2+1 \end{bmatrix}$$

4.a) Write down one step of the Jacobi and Gauss-Seidel methods.

Let us introduce the notation

$$d_N = \frac{1}{2N^2+1}$$

For the Jacobi method we have:

$$\begin{aligned} x_1^{n+1} &= d_N(N^2 x_2^n + b_1) \\ x_2^{n+1} &= d_N(N^2(x_1^n + x_3^n) + b_2) \\ x_3^{n+1} &= d_N(N^2(x_2^n + x_4^n) + b_3) \\ &\vdots \\ x_N^{n+1} &= d_N(N^2 x_{N-1}^n + b_N) \end{aligned}$$

For Gauss Seidel, it is very similar:

$$\begin{aligned} x_1^{n+1} &= d_N(N^2 x_2^n + b_1) \\ x_2^{n+1} &= d_N(N^2(x_1^{n+1} + x_3^n) + b_2) \\ x_3^{n+1} &= d_N(N^2(x_2^{n+1} + x_4^n) + b_3) \\ &\vdots \\ x_N^{n+1} &= d_N(N^2 x_{N-1}^{n+1} + b_N) \end{aligned}$$

4.b) Write down the relation between the error at step $n+1$ as a function of the error at step n.

The error at step $n+1$ is $E^{n+1} = x^{n+1} - x$ where is x is the exact solution. The Jacobi method may be written as

$$x^{n+1} = -D^{-1}(L+U)(x^n + b)$$

so the exact solution x verifies

$$x = -D^{-1}(L+U)(x+b).$$

We thus obtain that

$$E^{n+1} = -D(L+U)(x^n+b) - (-D^{-1}(L+U)(x+b)) = -D^{-1}(L+U)E^n$$

4.c) Show directly that the Jacobi method converge. What happens when N becomes very large?

For the error vector E^n we have

$$\begin{aligned} E_1^{n+1} &= d_N N^2 E_2^n \\ E_2^{n+1} &= d_N N^2(E_1^n + E_3^n) \\ E_3^{n+1} &= d_N N^2(E_2^n + E_4^n) \\ &\vdots \\ E_N^{n+1} &= d_N N^2 E_{N-1}^n \end{aligned}$$

so we have

$$|E_k^n| \le \frac{2N^2}{2N^2+1} \max_j |E_j^n|$$

which implies

$$\max_j |E_j^{n+1}| \le \frac{2N^2}{2N^2+1} \max_j |E_j^n|$$

Since $\frac{2N^2}{2N^2+1}$ is strictly lower than one, it means that $\max_j |E_j^n|$ converges to zero, so the error goes to zero. When N gets bigger, however, the constant $\frac{2N^2}{2N^2+1}$ approaches one, so the Jacobi iteration converges more and more slowly as N increases.