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MA2501 Numeriske Metoder
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Training Assignment 9

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The purpose of that assignment is to better understand quadrature formulae.

This assignment has 5 tasks.

Exercise 1. Suppose we want to find the zeros of the polynomial:

$$P(x) = -1 + x - x^2 + x^3$$

1.a) Write down the corresponding companion matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

1.b) Remember that you can create matrices in Python using array, so for instance the code

```
M = array([[1.,2.],[3.,4.]])
```

would produce the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Use the function eigvals to compute the eigenvalues of the companion matrix of p. Check that these are indeed all the roots of the polynomial p.

The matrix is produced by

```
array([[0.,0.,1.],[1.,0.,-1.],[0.,1.,1.]])
```

The eigenvalues are

```
array([ -3.60822483e-16+1.j,  -3.60822483e-16-1.j,
                  1.00000000e+00+0.j
                ])
```

which is approximately

$$\mathrm{i}, -\mathrm{i}, 1$$

and it can indeed be checked that

$$(x-1)(x+\mathrm{i})(x-\mathrm{i}) = (x-1)(x^2+1) = x^3 - x^2 + x - 1 = P(x)$$

Exercise 2. Suppose we choose interpolation points in the interval $[0,1]$:

$$c_0 = \frac{1}{4} \qquad c_1 = \frac{1}{2} \qquad c_2 = \frac{3}{4}$$

Remember that we obtain a quadrature formula by interpolating a function f at those points, and by integrating exactly the resulting interpolation polynomial. The resulting formula takes the form

$$\int_0^1 f(x)\,\mathrm{d}x \approx I(f) = \sum_{k=1}^{3} w_k f(c_k).$$

2.a) Explain why that quadrature formula integrates exactly the polynomials up to degree 2. Does it depend on the choice of c_0, c_1, c_2?

When interpolating a polynomial of degree up to 2 at three distinct point, one obtains the same polynomial back, so the integration formula is going to be exact for those.

2.b) Using the preceding fact on the polynomials 1, $x - \frac{1}{2}$, and $(x-\frac{1}{2})^2$ to find directly the weights w_k.

The formula has to integrate exactly the polynomials 1, $x - 1/2$ and $(x-1/2)^2$. From the second assertion we obtain $w_1 - w_3 = 0$. The last one gives

$$\int_0^1 (x-1/2)^2\,\mathrm{d}x = 1/24 = 1/16(w_1 + w_3) = w_1/8$$

This gives

$$w_1 = \frac{2}{3}$$

We know that $w_3 = w_1$, so

$$w_3 = \frac{2}{3}$$

Finally, from the assertion that constants are integrated exactly, we obtain $w_1 + w_2 + w_3 = 1$, so $w_2 = -\frac{1}{3}$.

We have thus obtained

$$w_1 = \frac{2}{3} \qquad w_2 = -\frac{1}{3} \qquad w_3 = \frac{2}{3}.$$

2.c) Write down the Lagrange polynomials ℓ_0, ℓ_1, ℓ_2 for the interpolation points c_k and compute their integrals $w_k = \int_0^1 \ell_k(x)\,\mathrm{d}x$. Do you find the same values of w_k? Explain why.

For instance, $\ell_0(x) = \frac{(x-1/2)(x-3/4)}{(-1/4)(-1/2)}$ which gives

$$\int_0^1 \ell_0(x)\,\mathrm{d}x = 8\int_0^1 (x^2 - \frac{5}{4}x + \frac{3}{8}) = 8(\frac{1}{3} - \frac{5}{8} + \frac{3}{8}) = \frac{8}{12} = \frac{2}{3}$$

A similar computation would yield the same confirming results for w_1 and w_2. The reason is that is that the quadrature formula is exact for $\ell_0(x)$, since ℓ_0 has degree 2. Now, using that $\ell_0(c_0) = 1$ and $\ell_0(c_1) = 0$ and $\ell_0(c_2) = 0$, the quadrature formula for ℓ_0 is just w_0, so we get

$$w_0 = \int_0^1 \ell_0(x)\,\mathrm{d}x$$

The same holds for w_1 and w_2.

2.d) Show that the quadrature formula integrates exactly polynomial of degree 3, but not 4. (Hint: use the quadrature formula on the polynomials $(x - \frac{1}{2})^3$ and $(x - \frac{1}{2})^4$)

It is straightforward to check that the formula is exact for the polynomial $(x - 1/2)^3$, and since it is exact for all polynomials for degree lower than or equal to two, it is exact for all the polynomials of degree lower than or equal to three.

For the polynomial $(x - 1/2)^4$ we have

$$\int_0^1 (x - 1/2)^4 = \frac{1}{5}\frac{1}{2^4} \neq 2\frac{2}{3}\left(\frac{1}{4}\right)^4 = \frac{1}{3}\frac{1}{2^6}$$

so the formula does not integrate exactly polynomials of degree 4.

2.e) Show that the quadrature formula is exact for the polynomial $(x - \frac{1}{2})^5$. Does that mean that it is exact for all the polynomials of degree 5?

Applying the quadrature formula on $(x - 1/2)^5$, one obtains zero, which is also the integral of $(x - 1/2)^5$, *but* that does not mean that all the polynomials of degree 5 are integrated exactly, since we do not integrate exactly all the polynomials of degree 4.

2.f) Write the quadrature formula scaled to an arbitrary interval $[a, b]$, in order to approximate $\int_a^b f(x)\,\mathrm{d}x$ (when the interval $[a, b]$ is small)

The scaled quadrature formula is

$$\int_a^b f(x)\,\mathrm{d}x \approx (b-a)\Big(\frac{2}{3}f((3a+b)/4) - \frac{1}{3}f((a+b)/2) + \frac{2}{3}f((a+3b)/4)\Big)$$

Exercise 3. We construct a quadrature formula using n points $c_0, \ldots, c_{n-1}$. Show that it is impossible to integrate exactly all polynomials of degree $2n$. (Hint: use the quadrature formula on the polynomial M^2, where $M(x) = (x - c_0)\cdots(x - c_{n-1})$.)

The quadrature formula applied to M^2 gives 0, but the integral of M^2 is not zero since M^2 is positive and not identically zero.

Exercise 4. **4.a)** Find two reals c_0, c_1 in $[0, 1]$ such that

$$\int_0^1 (x - c_0)(x - c_1) = 0 \tag{1}$$

We have one degree of freedom. Let us choose for instance $c_0 = 0$. Solving the equation $\int_0^1 x(x - c_1)\,\mathrm{d}x = 0$ we obtain $c_1 = 2/3$.

4.b) Compute the corresponding weights w_0, w_1

Ensuring that the quadrature formula $w_0 f(c_0) + w_1 f(c_1)$ integrates exactly constants, we obtain the equation

$$w_0 + w_1 = 1.$$

Ensuring that the formula is exact for first order polynomials, as x, we obtain

$$w_1 c_1 = \int_0^1 x\,\mathrm{d}x = 1/2$$

so

$$w_1 = \frac{3}{4}$$

and w_0 is thus

$$w_0 = \frac{1}{4}.$$

4.c) Show that the corresponding quadrature formula integrates exactly polynomials of degree 2.

Let us try the formula on x^2:

$$\frac{3}{4}\left(\frac{2}{3}\right)^2 = \frac{1}{3}$$

So the formula is indeed exact for polynomials of degree 2!

4.d) Show that this is in fact true for any choice of c_0 and c_1 as long as (1) is fulfilled. (Hint: use that $x^2 = (x - c_0)(x - c_1) + (c_0 + c_1)x - c_0c_1$ and the fact that a quadrature formula with two points will always integrate exactly polynomials of degree up to one)

Using the hinted expression, we obtain that $Q(x^2)$, the quadrature formula applied on the function x^2 is equal to

$$Q(x^2) = \int_0^1 (c_0 + c_1)x - c_0c_1 \,\mathrm{d}x$$

because the quadrature formula integrates exactly polynomials of degree one, and because it is zero for the function $(x - c_0)(x - c_1)$. Now we use (1) and obtain:

$$Q(x^2) = \int_0^1 (c_0 + c_1)x - c_0c_0 \,\mathrm{d}x + \underbrace{\int_0^1 (x - c_0)(x - c_1) \,\mathrm{d}x}_{=0}$$

from which we obtain:

$$Q(x^2) = \int_0^1 x^2 \,\mathrm{d}x$$

so the formula is exact for x^2. Since it was exact for polynomials of degree up to one, it is exact for all the polynomials of degree up to two.

Exercise 5. Suppose that we choose an odd number of interpolation points c_k for $k = 0, \dots, 2n$. Suppose further that the points are symmetrically placed around $\frac{1}{2}$, i.e., that

$$c_{n-k} + c_{n+k} = 1 \qquad \text{for} \qquad k = 0, \dots, n.$$

5.a) Show that the weights are symmetric around 1/2, i.e., that

$$w_{n-k} = w_{n+k} \qquad \text{for} \qquad k = 0, \dots, n.$$

We know that $w_{n-k} = \int_0^1 \ell_{n-k}(x)\,\mathrm{d}x$, where ℓ_k is the Lagrange polynomial for the node c_k.

5.b) Show that the quadrature formula integrates exactly polynomials of degree up to $2n + 1$.