Norwegian University of Science and Technology
Department of Mathematical Sciences

MA2501 Numeriske Metoder
Olivier Verdier

Training Assignment 10

2012-03-22

This assignment has 4 tasks.

Exercise 1. Suppose that we construct a quadrature formula (with nodes c_i and weights w_i) The corresponding integration formula is thus

$$I_h(f) = \sum_{k=0}^{N-1} h \sum_{i=1}^{s} w_i f(a_k + c_i h),$$

where $a_0 = a$, $a_N = b$, and $h = (b-a)/N$. Suppose that that the quadrature formula does not integrate constants exactly, i.e.,

$$\sum_{i=1}^{s} w_i \neq 1.$$

Show that the integration formula does not converge to the integral of f, i.e., in general

$$\lim_{h\to 0} I_h(f) \neq \int_a^b f(x)\,\mathrm{d}x.$$

Is that in agreement with the order formula derived in the lecture?

Define

$$C' := \sum_{i=1}^{s} w_i.$$

Choose the function $f = 1$, constant on $[a, b]$. Then

$$\int_a^b f(x)\,\mathrm{d}x = b - a$$

but the integration formula gives

$$I_h(1) = \sum_{k=0}^{N-1} h \sum_{i=1}^{s} w_i = C' \sum_{k=0}^{N-1} = C'(b-a) \neq b - a$$

Exercise 2. 2.a) Recall what the Vandermonde matrix is and what it was used for

2.b) Choose quadrature nodes $c_1, \ldots, c_s$ in the interval $[0, 1]$. The corresponding weights are chosen such that the quadrature is exact for polynomials of degree $s - 1$. Show that the vector w containing the corresponding weights, i.e., $w = (w_1, \ldots, w_s)$, is the solution of the linear system

$$V^{\mathsf{T}} w = b,$$

where the vector b is

$$b = (1, 1/2, \ldots, 1/s)$$

and V is the Vandermonde matrix for the points $c_1, \ldots, c_s$.

Each condition, for $k = 0, \ldots, s - 1$ reads

$$\sum_{i=1}^{s} c_i^k w_k = \int_0^1 x^k\,\mathrm{d}x = \frac{1}{k+1}$$

Exercise 3. 3.a) Compute the first three Legendre polynomials p_0, p_1 and p_2, by orthogonalising the polynomials 1, x, x^2 with respect to the scalar product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,\mathrm{d}x$$

Starting with $\tilde{p}_0(x) = 1$, we only have to normalize it:

$$p_0 = 1/2$$

Now

$$\tilde{p}_1(x) = a + x$$

By writing $\langle \tilde{p}_1, p_0 \rangle = 0$ one obtains $a = 0$. The normalisation condition is given by

$$\langle \tilde{p_1}, \tilde{p_1} \rangle = \int_{-1}^{1} x^2 = 1 = \frac{2}{3}$$

So we take

$$p_1(x) = \sqrt{\frac{3}{2}}x$$

The next polynomial would be

$$\tilde{p}_2(x) = a + bx + x^2$$

The orthogonal condition $\langle \tilde{p}_2, p_0 \rangle = 0$ becomes

$$a + 1/3 = 0$$

and the orthogonal condition $\langle \tilde{p}_2, p_1 \rangle = 0$ yields

$$b = 0$$

We deduce that

$$\tilde{p}_2(x) = -\frac{1}{3} + x^2$$

We compute

$$\langle \tilde{p}_2, \tilde{p}_2 \rangle = 2\Big(\frac{1}{9} - \frac{2}{9} + \frac{1}{5}\Big) = \frac{8}{45}$$

so

$$p_2(x) = \sqrt{\frac{45}{8}}(-\frac{1}{3} + x^2)$$

3.b) Compute the roots of the polynomial p_2. Compute the weights of the corresponding formula (hint: apply the formula to the polynomial 1 and x that are integrated exactly)

The roots of p_2 are given by

$$r = \pm\frac{1}{\sqrt{3}}$$

The corresponding integration formula is

$$I(f) = w_1 f(-1/\sqrt{3}) + w_2 f(1/\sqrt{3})$$

We now use the fact that I integrates exactly the polynomials 1 and x between -1 and 1.

We thus know that $I(1) = \int_{-1}^{1} 1 \, \mathrm{d}x = 2$, which leads to

$$I(1) = w_1 + w_2 = 2$$

We know that $I(x) = \int_{-1}^{1} x \, \mathrm{d}x = 0$, so

$$I(x) = w_1 \frac{-1}{\sqrt{3}} + w_2 \frac{1}{\sqrt{3}} = 0$$

from which we deduce that $w_1 = w_2$. The weights are thus $w_1 = w_2 = 1$ and the integration formula is thus

$$I(f) = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

3.c) Check that this integration formula integrates exactly polynomials of degree lower than or equal to 3, but not 4.

3.d) What is the order of the corresponding integration formula?

We already know that the formula integrates exactly polynomials of degree ≤ 1, so we only need to check the claim for x^2 and x^3.

$$I(x^2) = 1/3 + 1/3 = 2/3 = \int_{-1}^{1} x^2$$

so x^2 is integrated exactly. For the polynomial x^3 we have $I(x^3) = 0$ by symmetry of the coefficients, and we also have $\int_{-1}^{1} x^3 = 0$, so x^3 is indeed integrated exactly.

Exercise 4. There is a recursion relation between the Legendre polynomials, the goal is to find it out.

4.a) Show that the polynomial xp_k is orthogonal to all the polynomials of degree less than or equal to $k - 2$.

4.b) Expand xp_k in the basis $p_0, \ldots, p_{k+1}$ to find the recurrence relation

Suppose that the polynomials are orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_I fg$$

where I is a given interval (for instance $[0, 1]$). We have

$$A = \langle xp_k, p_j \rangle = \int_I xp_k p_j = \int_I p_k (xp_j)$$

Now notice that the degree of xp_j is $j + 1$, so if $j \leq k - 2$, the degree of xp_j is lower or equal than $k - 1$, so the quantity A is zero by orthogonality of p_k with all the polynomials of lower degree.

To see the recurrence relation, we may expand the polynomial xp_k on the orthogonal basis $p_0, p_1, \ldots, p_{k+1}$ from which we obtain

$$xp_k = \langle xp_k, p_{k+1} \rangle\, p_{k+1} + \langle xp_k, p_k \rangle\, p_k + \langle xp_k, p_{k-1} \rangle\, p_{k-1}$$

Observe that $\langle xp_k, p_{k+1} \rangle \neq 0$, so the recursion relation takes the form

$$p_{k+1} = \alpha_k x p_k + \beta_k p_k + \gamma_k p_{k-1}$$