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MA2501 Numeriske Metoder
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Training Assignment 11

2012-03-29

The purpose of those exercises is to become familiar with the discrete Fourier transform and its corresponding algorithm, the fast Fourier transform.

Some useful formulae and definitions:

$$\omega_N := \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}}$$

The Fourier matrix of size N is defined as

$$F_N = [\omega_N^{ij}]_{i,j=0,\ldots,N-1}$$

The discrete Fourier transform of $z = (z_0, \ldots, z_{N-1})$ is defined by

$$y_k = \sum_{j=0}^{N-1} z_k \omega_N^{kj},$$

so in matrix vector notation, this is simply

$$y = F_N z.$$

This assignment has 5 tasks.

Exercise 1. 1.a) Compute ω_2, ω_4 and ω_8.

$$\omega_2 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{2}} = \mathrm{e}^{\mathrm{i}\pi} = -1$$
$$\omega_4 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{4}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$$
$$\omega_8 = \mathrm{e}^{\mathrm{i}\frac{2\pi}{8}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$$

1.b) Compute ω_8^8, ω_8^9, and more generally, ω_N^{N+1}.

$$\omega_8^8 = (e^{i\frac{2\pi}{8}})^8 = e^{i2\pi} = 1$$
$$(\omega_8)^9 = (\omega_8^8)\omega_8 = \omega_8$$
$$\omega_N^{N+1} = (\omega_N^N)\omega_N = \omega_N$$

The last equality is due to

$$\omega_N^N = e^{i\frac{2\pi}{N}N} = e^{i2\pi} = 1$$

1.c) Compute ω_{2n}^n

$$\omega_{2n}^n = e^{i\frac{2\pi n}{2n}} = e^{i\pi} = -1$$

Exercise 2. **2.a)** Write down the matrices F_2 and F_4.

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

To compute F_4 we first notice that $\omega_4 = i$. Now, by definition

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

and by using that $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$, we obtain

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

2.b) Compute the discrete Fourier transform of

$$z = (1, -1, 1, -1).$$

What do you notice? What is the explanation?

The result is simply

$$(0, 0, 4, 0)$$

It reflects the fact that the matrices F_N are orthogonal (in the complex sense). Now since z was the column number three of that matrix, it means that $F_N z$ must be zero for all the entries but the third one.

2.c) Create new two by two submatrices from F_4 by following the following prescriptions:

A_{00}**:** First two rows and even columns

A_{10}**:** First two rows and odd columns

A_{10}**:** Last two rows and even columns

A_{11}**:** Last two rows and odd columns

Express those submatrices from F_2 and from the matrix Ω_2 defined as

$$\Omega_2 := \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{bmatrix}.$$

We obtain the matrices

$$A_{00} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$A_{10} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$A_{01} = \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}$$
$$A_{11} = \begin{bmatrix} -1 & -1 \\ -\mathrm{i} & \mathrm{i} \end{bmatrix}$$

It is clear that $A_0 = F_2$ and that $A_1 = F_2$. We also check that $A_2 = \Omega_2 F_2$ and $A_3 = -\Omega_2 F_2$.

Exercise 3. Suppose we have a periodic function $f(x)$ which we want to approximate as a sum

$$f(\theta) = \sum_{k=0}^{N-1} a_k \cos\left(\frac{2\pi}{N} k\theta\right)$$

How would you use the discrete Fourier transform for that?

Exercise 4. Recall the fast Fourier transform formula, if

$$y = F_{2n} z,$$

then

$$y_j = \sum_{k=0}^{n-1} (z'')_k \omega_n^{kj} + \omega_{2n}^j \sum_{k=0}^{n-1} (z')_k \omega_n^{kj} \tag{1}$$

where z'' is composed of the even components of z, and z' is composed of the odd components of z, that is

$$(z'')_j \coloneqq z_{2j}, \qquad (z')_j \coloneqq z_{2j+1}.$$

4.a) Write z' and z'' for a vector $z = (7, 6, 5, 4, 3, 2, 1, 0)$.

The vector z'' is composed of the even components of z, *starting at zero*, so we have

$$z'' = (z_0, z_2, z_4, z_6)$$

which is just

$$z'' = (7, 5, 3, 1)$$

in that case. Similarly, the vector z' is construct as

$$z' = (z_1, z_3, z_5, z_7)$$

which gives

$$z' = (6, 4, 2, 0)$$

in that case.

4.b) Compute the formula (1) for $n = 2$. Make sure to group together the values $z''_0 + z''_1$, $z''_0 - z''_1$ and $z'_0 + z'_1$, $z'_0 - z'_1$.

In the case $n = 2$ we obtain after some simplification

$$\begin{aligned} y_0 &= z''_0 + z''_1 + z'_0 + z'_0 \\ y_1 &= z''_0 - z''_1 + \mathrm{i}(z'_0 - z'_1) \\ y_2 &= z''_0 + z''_1 - (z'_0 + z'_1) \\ y_3 &= z''_0 - z''_1 - \mathrm{i}(z'_0 - z'_1) \end{aligned}$$

One sees in those equations that one can first compute $z''_0 + z''_1$, $z''_0 - z''_1$ and $z'_0 + z'_1$, $z'_0 - z'_1$ first. Note that these are nothing else than the components of the "coarser" Fourier transform, namely $F_2 z''$ and $F_2 z'$.

4.c) Show that for $j = 0, \dots, n - 1$,

$$y_{j+n} = \sum_{k=0}^{2n-1} (z'')_k \omega_n^{kj} - \omega_{2n}^{j} \sum_{k=0}^{2n-1} (z')_k \omega_n^{kj}.$$

If we introduce $j + n$ in formula (1), we obtain:

$$y_{j+n} = \sum_{k=0}^{n-1} (z'')_k \omega_n^{k(j+n)} + \omega_{2n}^{j+n} \sum_{k=0}^{n-1} (z')_k \omega_n^{k(j+n)}$$

Now, notice that

$$\omega_n^{k(j+n)} = \omega_n^{kj} \underbrace{\omega_n^{kn}}_{=(\omega_n^n)^k=1^k=1} = \omega_n^{kj}$$
$$\omega_{2n}^{j+n} = \omega_{2n}^{j}\omega_{2n}^{n} = \omega_{2n}^{j}(-1) = -\omega_{2n}^{j}$$

which proves that the formula is correct.

Exercise 5. Use the Fast Fourier Transform to compute

$$F_4 z$$

where

$$z = (0, 1, 1, 0)$$

and check that the result is correct by computing the matrix multiplication $F_4 z$ directly.

Recall that the fast Fourier transform algorithm may be written as

$$y_1 = F_2 z'' + \Omega_2 F_2 z'$$
$$y_2 = F_2 z'' - \Omega_2 F_2 z'$$

where $y = (y_1, y_2)$, and the matrices F_2 and Ω_2 are given by

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{bmatrix}$$

Now, $z'' = (0, 1)$ and $z' = (1, 0)$, so

$$F_2 z'' = (1, -1) \qquad F_2 z' = (1, 1)$$

and

$$\Omega_2(F_2 z') = (1, \mathrm{i})$$

We thus obtain that

$$y_1 = (1, -1) + (1, \mathrm{i}) = (2, -1 + \mathrm{i})$$
$$y_2 = (1, -1) - (1, \mathrm{i}) = (0, -1 - \mathrm{i})$$

so

$$y = (2, -1 + \mathrm{i}, 0, -1 - \mathrm{i})$$

We compare that with the direct calculation of $F_4 z$ now:

$$y = (1 + 1, \mathrm{i} - 1, -1 + 1, -\mathrm{i} - 1)$$

and we see that we obtain the same result.