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MA2501 Numeriske Metoder
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Training Assignment 12

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This assignment has 4 tasks.

Exercise 1. Consider the following differential equation:

$$u'(t) = -u(t)\cos(t)$$

1.a) Try to understand what is the unknown in that equation. Write one possible initial condition.

The unknown is the function u itself. A possible initial condition would be $u(0) = 1.$, for instance.

1.b) Write down one step of the Explicit and Implicit Euler methods for that differential equation

We assume in both cases a time step Δt. For explicit Euler, we have

$$u_{n+1} = u_n - \Delta t u_n \cos(t_n)$$

For implicit Euler, the next value u_{n+1} is defined implicitly by:

$$u_{n+1} = u_n - \Delta t u_{n+1} \cos(t_n + \Delta t)$$

Exercise 2. Consider Newton's Equation modelling an oscillator without friction.

$$u''(t) = -u(t)$$

2.a) Write down one step of the Explicit Euler method to solve that equation numerically.

The trick is to first transform this equation into a first order differential equation. To this end, introduce a new variable (the "velocity") v which is equal to

$$v(t) = u'(t).$$

The original equation is equivalent to the equation

$$\begin{aligned} u'(t) &= v(t) \\ v'(t) &= -u(t) \end{aligned}$$

Now it is easy to discretise that equation with, for instance, the explicit Euler method. We obtain

$$\begin{aligned} u_{n+1} &= u_n + \Delta t\, v_n \\ v_{n+1} &= v_n - \Delta t\, u_n. \end{aligned}$$

2.b) Do the same with the Runge Kutta 4 method (p. 443 in C&K).

Exercise 3. Given a numerical method, for instance explicit Euler, one may define the corresponding "flow" as a mapping:

$$\Phi_h : \ u_0 \longmapsto u_1$$

For instance, in the explicit Euler case, this mapping is given by

$$\Phi_h(u_0) = u_0 + hf(u_0)$$

The *adjoint method* corresponding to a given flow is given by the flow

$$\Psi_h := (\Phi_{-h})^{-1}$$

3.a) What is the adjoint method corresponding to explicit Euler?

From

$$u_1 = u_0 - hf(u_0)$$

we deduce

$$u_0 + hf(u_0) = u_1$$

which is the *implicit* Euler method (obtaining u_0 from u_1)

3.b) What is the adjoint method of the Trapezoidal rule, given by:

$$\Phi_h(u_0) = u_1 \quad \text{such that} \quad u_1 - u_0 = h(f(u_0) + f(u_1))/2$$

One sees immediately that

$$\Phi_{-h}(u_0) = u_1 \iff u_0 = \Phi_h u_1$$

so $\Phi_{-h}^{-1} = \Phi_h$. The adjoint method of the Trapezoidal rule is thus the trapezoidal rule itself.

Exercise 4. In this exercise, we compute the stability region of the trapezoidal rule.

4.a) Apply the trapezoidal rule to the differential equation $u' = \lambda u$, and write the result as

$$u_1 = A(\lambda h) u_0$$

where A is an expression that you will compute.

Introducing $f(u) = \lambda u$ in the expression for the trapezoidal rule we obtain

$$u_1 = u_0 + h\lambda(u_0 + u_1)$$

from which we obtain

$$u_1 = \frac{1 + h\lambda}{1 - h\lambda} u_0$$

so

$$A(z) = \frac{1+z}{1-z}$$

4.b) What is the region of the complex plane corresponding to the equation

$$|A(z)| \leq 1$$

What restriction is there to the step size h if $\Re(\lambda) < 0$ (stable system)?

The condition is that

$$|A(z)| \leq 1$$

which is equivalent to

$$|A(z)|^2 \leq 1$$

which is equivalent to

$$|1 + z|^2 \leq |1 - z|^2$$

(because $|1 - z| \geq 0$) From this we obtain

$$1 + 2\Re z + |z|^2 \leq 1 - 2\Re z + |z|^2$$

which is equivalent to

$$\Re z \leq 0$$

The stability region is thus the whole half complex plane.

Assuming that $\Re\lambda \leq 0$, this implies that for any h, $\Re(h\lambda) \leq 0$ so the method is stable without any restriction on the stepsize h.