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TMA4310 Optimal control of PDEs
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Exercise set 1

1 We consider an optimal control problem for launching a rocket to a given altitude $H > 0$. We make simplifying assumptions that the mass $m > 0$ of the rocket stays constant during the flight and does not change as we burn fuel, ignore the change in gravity g with the altitude, and also the forces due to the friction with the atmosphere. The state of the system (flight plan, so to speak) is given by a function $y : \mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto y(t)$, which describes the altitude change with time, and the control $t \mapsto u(t)$ is the force produced by the engine. Thus the state equation is given by an initial value problem (Newton's 2nd law):

$$\begin{aligned} my''(t) &= u(t) - mg, \\ y(0) &= y'(0) = 0. \end{aligned} \tag{1}$$

We would like to minimize the fuel needed to attain the altitude H (we assume that the rate of fuel consumption is proportional to the exerted force $u(t)$):

$$J(y,u) = \int_0^T |u(t)|\,\mathrm{d}t,$$

where $T > 0$ is the time at which the rocket reaches H. Note that T is not a constant but depends on y and u.

a) Integrate the state equation (1) twice (we assume that the control u is regular enough to allow us to do this) to show that

$$y(t) = \frac{1}{m}\int_0^t (t-\tau)u(\tau)\,\mathrm{d}\tau - \frac{gt^2}{2}. \tag{2}$$

In particular, T satisfies the equation

$$H + \frac{gT^2}{2} = \frac{1}{m}\int_0^T (T-\tau)u(\tau)\,\mathrm{d}\tau. \tag{3}$$

b) Utilize Hölder's inequality in the right hand side of (3) to show the following lower bound on J for all controls/states satisfying (2):

$$J(y,u) \geq m\left(\frac{H}{T} + \frac{gT}{2}\right). \tag{4}$$

Further argue that the inequality is *strict* for all functions u. (The equality can only be attained when u is a distribution.)

c) Note that the right hand side of (4) still depends on (y,u) through T. Show that the following bound holds for all $T > 0$:

$$m\left(\frac{H}{T} + \frac{gT}{2}\right) \geq m\sqrt{2gH}. \tag{5}$$

Combining the *strict* bound (4) and the greater than or equal bound (5) we can see that the lower bound $m\sqrt{2gH}$ is not attained by any control. We will now construct a sequence of controls/states (y_n, u_n) such that $\lim_{n\to\infty} J(y_n, u_n) = m\sqrt{2gH}$. The underlying idea is that (3) implies that burning fuel "early" counts with a larger weight towards attaining the altitude H.

Let us consider the following control:

$$u_n(t) = \begin{cases} \alpha_n, & 0 < t < n^{-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $\alpha_n > 0$ is a constant to be determined.

d) Calculate the value of α_n from (3). Note: you will end up with a quadratic equation for T (i.e., the rocket reaches T once on the way up, and once on the way down when it starts falling due to the gravity pulling.) Select the value of α_n such that the quadratic equation admits only one root, i.e., we burn the smallest amount of fuel so that the rocket reaches the altitude H with velocity 0 and then starts falling down.

e) Show that $\lim_{n\to\infty} J(y_n, u_n) = m\sqrt{2gH}$. Combined with the *strict* bound (4) and the greater than or equal bound (5), this shows that the original optimal control problem does not admit an optimal solution in the class of regular functions. However, we can approximate the lower bound arbitrarily closely by for example using the controls of the type (6).

$\boxed{2}$ We consider a(n artificial) finite-dimensional optimal control problem for $y \in \mathbb{R}^2$ with a control parameter $u \in \mathbb{R}$.

The state equation is:

$$\begin{aligned} y_1 + y_2 &= u, \\ y_2 &= 2u, \end{aligned} \tag{7}$$

and the const functional is

$$J(y,u) = \frac{1}{2}[(y_1 - 1)^2 + (y_2 - 2)^2] + \frac{\lambda}{2}u^2, \tag{8}$$

where $\lambda > 0$.

a) Derive the explicit expressions for the reduced cost functional and its gradient.

b) Formulate the adjoint problem and compute the reduced gradient with the help of the adjoint state.

c) Assuming $U_{\text{ad}} = \mathbb{R}$ state the first order necessary optimality conditions for this problem.