Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4310 Optimal control of PDEs
Spring 2015

Exercise set 3

This week we will study additional, slightly more technical, properties of Sobolev functions: most notably the possibility of approximation with smooth functions, extensions, and existence of traces (well defined values on the boundary).

Reading:

Reading: more details about Sobolev spaces (Sections 5.3-5.6.2 in [Evans]). Skim through Section 5.6.3.

Recommended exercises:

- Let $U = B(0,1) \subset \mathbb{R}^2$, i.e. the unit ball in $\mathbb{R}^2$. Make a partition of the boundary into a few overlapping subsets (say, three or four) and define in a neighbourhood of one of the subsets a mapping $\Phi(x)$ that straightens the boundary (see [Evans, Appendix C.1 or the illustrations in Section 5.4]).
- [Evans], Chapter 5: 14, 2, 4.
- Often steady-state solutions to the incompressible Navier–Stokes equations are found using the following fixed point type iteration. At the step k we know the approximation $\vec{u}^k = (u_1^k, u_2^k, u_3^k) \in [W_0^{1,2}(U)]^3]$ and a function $p^k \in L^2(U)$. We look for the next approximation by solving the Stokes problem:

$$-\sum_{\alpha=1}^{3} D^2_{\alpha,\alpha} u_i^{k+1} + D_i p^{k+1} = f_i - \sum_{\alpha=1}^{3} u_\alpha^k D_\alpha u_i^k,$$
$$\sum_{\alpha=1}^{3} D_\alpha u_\alpha^{k+1} = 0,$$

  where $(f_1, f_2, f_3) \in [L^2(\Omega)]^3$ is a given function. The Stokes problem admits a unique weak solution when the right hand side of the first (momentum conservation) vector equation is a bounded linear functional on $[W^{1,2}(U)]^3$. Show that this is indeed the case. That is, show that

$$L_i^k(v) = \int_U \left[f_i(x) - \sum_{\alpha=1}^{3} u_\alpha^k D_\alpha u_i^k(x) \right] v(x)\, \mathrm{d}x,$$

  is a bounded linear functional on $W^{1,2}(U)$. Hint: Gagliardo–Nirenberg–Sobolev + Hölder inequalities.

- Exercise 10 from Chapter 5 in [Evans]. What can we say when U has several connected components?

  Hint: apply Gagliardo–Nirenberg–Sobolev inequality sufficiently many times; then Morrey's inequality should be applicable. Utilize the modification of Morrey's inequality discussed in the Remark at the end of Section 5.6.2.

- Provide the details of the proof to the modification of Morrey's inequality discussed at the end of Section 5.6.2.