TMA4310 Optimal control of PDEs
Spring 2015

Exercise set 4

We can now establish existence and uniqueness of weak solutions to a wide range of elliptic partial differential equations.

Reading:

Sections 5.7, 5.8.1, 5.8.3; 6.1.1–6.2.1 in [Evans]).

Recommended exercises:

1. Consider a sequence $\{u_k\}$ in $W^{1,p}(U)$, where U is an open set in $\mathbb{R}^n$. Assume that $\{u_k\}$ converges weakly in $W^{1,p}(U)$ to a limit $\hat{u} \in W^{1,p}(U)$. Show that $\{u_k\}$ also converges weakly in $L^p(U)$ to $\hat{u}$, and that $\{D_{x_j} u_k\}$ converges weakly in $L^p(U)$ towards $D_{x_j}\hat{u}$, $1 \le j \le n$.

2. Establish the following Poincare-type inequality: let $U \subset \mathbb{R}^n$ be a non-empty connected bounded set with C^1 boundary ∂U, $1 \le p < +\infty$, and let $T : W^{1,p}(U) \to L^p(\partial U)$ be the trace operator. Show that there is a positive constant C such that for every $u \in W^{1,p}(U)$ we have the estimate
$$\|u - (u)_{\partial U}\|_{L^p(U)} \le C\|Du\|_{L^p(U)},$$
where $(u)_{\partial U}$ is the average trace of u:
$$(u)_{\partial U} = \frac{1}{|\partial U|}\int_{\partial U} [Tu](x)\,\mathrm{d}x.$$
Hint: modify the proof of [Theorem 1, Section 5.8.1, Evans] as necessary. Use the fact that the constructed sequence $\{v_{k_j}\}$ is Cauchy in $W^{1,p}(U)$, and utilize Exercise 10 (Chapter 5; last week's exercies) to characterize its limit.

3. Use the previous exercise in conjunction with Lax–Milgram theorem to show that Laplace equation with homogeneous Dirichlet boundary condition admits a unique weak solution.

4. Consider the remark after the proof of [Theorem 1, Section 6.2.1, Evans] (Lax–Milgram Theorem). Provide the details of the simpler proof of Lax–Milgram theorem for symmetric bilinear forms sketched in the remark.

5. Establish the following generalization of Lax–Milgram theorem. Let U, V be two Hilbert spaces and consider a bilinear form B on $U \times V$, satisfying the following conditions:

   (a) Continuity: there is a constant $\alpha > 0$ such that $B[u, v] \le \alpha\|u\|_U\|v\|_V$.

(b) Inf-sup, or Ladyzhenskaya–Babuška–Brezzi (LBB) condition: there is a constant $\beta > 0$ such that
$$\inf_{u \in U \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{B[u,v]}{\|u\|_U \|v\|_V} \geq \beta.$$

(c) For all $v \in V \setminus \{0\}$ it holds that $\sup_{u \in U} |B(u,v)| > 0$.

Then for every bounded linear functional f on V the variational problem

$$B[u,v] = f(v), \qquad \forall v \in V,$$

admits a unique solution $\hat{u} \in U$. This solution satisfies the estimate $\|\hat{u}\|_U \leq \beta^{-1} \|f\|_{V'}$.

Sketch of the proof:

(a) Show that the functional $F_u : V \to \mathbb{R}$ defined by $F_u(v) = B[u,v]$ is bounded and linear.

(b) Define an operator $A : U \to V$ by $Au = R_V^{-1}(F_u)$, where $R_V : V \to V'$ is the Riesz' map $V \ni v \mapsto (v, \cdot)_V \in V'$. Show that this operator is linear and bounded.

(c) Use LBB condition to show that A is bounded from below, that is $\|Au\|_V \geq \beta \|u\|_U$.

(d) From the previous condition show that A admits a bounded inverse A^{-1} acting from the range $R(A) \subset V$ of A to U.

(e) Again using the boundedness of A from below, show that $R(A)$ is closed in V.

(f) Use the last assumption of the theorem to show that $R(A)^{\perp} = \{0\}$; therefore $R(A) = V$.

(g) Conclude the proof of existence by showing that $\hat{u} = A^{-1} R_V^{-1} f$ solves the variational problem.

(h) The norm bound on the solution follows from the boundedness of A^{-1}. The uniqueness of the solution follows from this bound and the linearity.