Norwegian University of Science and Technology
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TMA4310 Optimal control of PDEs
Spring 2015

Exercise set 5

We can now establish existence of optimal controls for linear-quadratic control problems of elliptic PDEs, and move towards the optimality conditions for these problems.

Reading:

Section 2.4.2-2.7 in [Tr].

Recommended exercises:

1. Prove Theorem 2.12 in [Tr] using Theorem 2.11. Hint: for any $\alpha \in \mathbb{R}$, the set $\{ v \in U \mid f(v) \leq \alpha \}$ is convex and closed provided that f is convex and continuous (in fact, even lower-semicontinuity is sufficient). Assume that $f(u) > \liminf_{n\to\infty} f(u_n) + \epsilon$ for some $\epsilon > 0$, and show that there must be a subsequence $u_{n_k} \in \{ v \in U \mid f(v) \leq f(u) - \epsilon/2 \}$.

2. Exercise 2.9 in [Tr]. Also show that the set considered in this exercise has an empty interior (in particular this prevents the application of standard optimization results about the necessary optimality conditions).

3. Consider the control problem (2.26)–(2.28) in [Tr], and the associated solution operatir $S : L^2(\Omega) \to L^2(\Omega)$. As a consequence of Lax–Milgram theorem, this operator is continuous.

   (a) Utilize Rellich–Kondrachov's theorem to conclude that this operator is in fact *compact* (it transforms weakly converging sequences into strongly convergent sequences).

   (b) As a consequence, show that the quadratic functional $f(u)$ in Theorem 2.14, [Tr] can be replaced with a much more general functional

   $$f(u) = f_1(Su) + f_2(u),$$

   where $f_1 : H \to \mathbb{R}$ is any continuous (not necessarily convex) function and $f_2 : U \to \mathbb{R}$ is any continuous and *convex* function of u, without changing the conclusion of Theorem 2.14.

4. Exercise 2.11 in [Tr].

5. Exercises 2.7, 2.8 in [Tr].