## **Problems**

- 1. Let S = C[0,1] be the set of real-valued continuous functions defined on the closed interval [0,1], where we define f + g and fg, as usual, by (f+g)(x) = f(x) + g(x) and (fg)(x) = f(x)g(x). Let 0 and 1 be the constant functions 0 and 1, respectively. Show that
  - (a)  $(S, +, \cdot)$  is a commutative ring with unity.
  - (b) S has nonzero zero divisors.
  - (c) S has no idempotents  $\neq 0,1$ .
  - (d) Let  $a \in [0,1]$ . Then the set  $T = \{ f \in S | f(a) = 0 \}$  is a subring such that fg,  $gf \in T$  for all  $f \in T$  and  $g \in S$ .
- 2. Let R be an integral domain and  $a,b \in R$ . If  $a^m = b^m$ ,  $a^n = b^n$ , and (m,n) = 1, show that a = b.
- 3. (a) Show that the following are subrings of C.

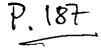
  - (i) A = {a + b√-1 | a,b ∈ Z}.
    (ii) B = (a + b√-3 | either a,b ∈ Z or both a,b are halves of odd integers).

The set A is called the ring of Gaussian integers.

- (b) Let e be an idempotent in a ring R. Show that the set eRe = $(eae|a \in R)$  is a subring of R with unity e.
- 4. Show that an integral domain contains no idempotents except 0 and 1 (if 1 exists).
- 5. (a) Determine the idempotents, nilpotent elements, and invertible elements of the following rings:
  - (i)  $\mathbb{Z}/(4)$ (ii)  $\mathbb{Z}/(20)$
  - (b) Show that the set U(R) of units of a ring R with unity forms a multiplicative group (cf. Problem 18(c)).
  - (c) Prove that an element  $\bar{x} \in \mathbb{Z}/(n)$  is invertible if and only if (x,n) = 1. Show also that if (x,n) = 1, then  $x^{\phi(n)} \equiv 1 \pmod{n}$ , where  $\phi(n)$  is Euler's function (this is called the Euler-Fermat theorem).
- 6. Let S be the set of  $2 \times 2$  matrices over Z of the form  $\binom{a}{b}$ . Show that
  - (a) S is a ring.
  - (b)  $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^k = \begin{pmatrix} a^k & * \\ 0 & c^k \end{pmatrix}$ , where \* denotes some integer.

Also find the idempotents and nilpotent elements of S. Show that nilpotent elements form a subring.

If a and b are nilpotent elements of a commutative ring, show that a+b is also nilpotent. Give an example to show that this may fail if the ring is not commutative.



## **Problems**

Let R be a commutative ring with unity. Suppose R has no nontrivial ideals. Prove that R is a field.

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## **Problems**

1. Find the ideals of the ring  $\mathbb{Z}/(n)$ .

2. Prove that  $\mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i]$ , where  $\mathbb{Z}[i] = \{a+b\sqrt{-1}|a,b\in\mathbb{Z}\}.$ 

3. Show that there exists a ring homomorphism  $f: \mathbb{Z}/(m) \to \mathbb{Z}/(n)$  sending  $\overline{1}$  to  $\overline{1}$  if and only if n|m.

- 4. Show that the set N of all nilpotent elements in a commutative ring R forms an ideal. Also show that R/N has no nonzero nilpotent elements. Give an example to show that N need not be an ideal if R is not commutative.
- 5. Let S be a nonempty subset of R. Let

$$r(S) = \{x \in R | Sx = 0\}$$
 and  $l(S) = \{x \in R | xS = 0\}.$ 

Then show that r(S) and l(S) are right and left ideals, respectively [r(S) and l(S) are called right and left annihilators of S, respectively].

- 6. In Problem 5 show that r(S) and l(S) are ideals in R if S is an ideal in R.
- 7. Show that any nonzero homomorphism of a neld F into a ring R is 1-1.