P. 173 - 174

Problems

1. Let $S = C[0,1]$ be the set of real-valued continuous functions defined on the closed interval $[0,1]$, where we define $f+g$ and fg, as usual, by $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. Let 0 and 1 be the constant functions 0 and 1, respectively. Show that
   (a) $(S, +, \cdot)$ is a commutative ring with unity.
   (b) S has nonzero zero divisors.
   (c) S has no idempotents $\neq 0, 1$.
   (d) Let $a \in [0,1]$. Then the set $T = \{f \in S | f(a) = 0\}$ is a subring such that $fg, gf \in T$ for all $f \in T$ and $g \in S$.
2. Let R be an integral domain and $a,b \in R$. If $a^m = b^m$, $a^n = b^n$, and $(m,n) = 1$, show that $a = b$.
3. (a) Show that the following are subrings of **C**.
   (i) $A = \{a + b\sqrt{-1} | a,b \in \mathbf{Z}\}$.
   (ii) $B = \{a + b\sqrt{-3} |$ either $a,b \in \mathbf{Z}$ or both a,b are halves of odd integers$\}$.

   The set A is called the ring of *Gaussian integers*.
   (b) Let e be an idempotent in a ring R. Show that the set $eRe = \{eae | a \in R\}$ is a subring of R with unity e.
4. Show that an integral domain contains no idempotents except 0 and 1 (if 1 exists).
5. (a) Determine the idempotents, nilpotent elements, and invertible elements of the following rings:
   (i) $\mathbf{Z}/(4)$ (ii) $\mathbf{Z}/(20)$
   (b) Show that the set $U(R)$ of units of a ring R with unity forms a multiplicative group (cf. Problem 18(c)).
   (c) Prove that an element $\bar{x} \in \mathbf{Z}/(n)$ is invertible if and only if $(x,n) = 1$. Show also that if $(x,n) = 1$, then $x^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n)$ is Euler's function (this is called the *Euler–Fermat theorem*).
6. Let S be the set of 2×2 matrices over **Z** of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. Show that
   (a) S is a ring.
   (b) $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^k = \begin{pmatrix} a^k & * \\ 0 & c^k \end{pmatrix}$, where $*$ denotes some integer.

   Also find the idempotents and nilpotent elements of S. Show that nilpotent elements form a subring.
7. If a and b are nilpotent elements of a commutative ring, show that $a + b$ is also nilpotent. Give an example to show that this may fail if the ring is not commutative.

P. 187

Problems

1. Let R be a commutative ring with unity. Suppose R has no nontrivial ideals. Prove that R is a field.

P. 194 - 195

Problems

1. Find the ideals of the ring $\mathbf{Z}/(n)$.
2. Prove that $\mathbf{Z}[x]/(x^2+1) \simeq \mathbf{Z}[i]$, where $\mathbf{Z}[i] = \{a + b\sqrt{-1} \mid a,b \in \mathbf{Z}\}$.
3. Show that there exists a ring homomorphism $f: \mathbf{Z}/(m) \to \mathbf{Z}/(n)$ sending $\bar{1}$ to $\bar{1}$ if and only if $n \mid m$.
4. Show that the set N of all nilpotent elements in a commutative ring R forms an ideal. Also show that R/N has no nonzero nilpotent elements. Give an example to show that N need not be an ideal if R is not commutative.
5. Let S be a nonempty subset of R. Let

   $$r(S) = \{x \in R \mid Sx = 0\} \quad \text{and} \quad l(S) = \{x \in R \mid xS = 0\}.$$

   Then show that $r(S)$ and $l(S)$ are right and left ideals, respectively [$r(S)$ and $l(S)$ are called right and left *annihilators* of S, respectively].
6. In Problem 5 show that $r(S)$ and $l(S)$ are ideals in R if S is an ideal in R.
7. Show that any nonzero homomorphism of a field F into a ring R is 1-1.