MA3201 Rings and Modules, 2014

Notes on Differential Equations

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a vector function of t. Then $\dot{\mathbf{x}} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$. Consider the first order linear differential equation with initial conditions:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x}; \\ \mathbf{x}(0) &= \mathbf{x}_0. \end{aligned} \tag{1}$$

Given an $n \times n$ matrix A over $\mathbb{R}$, define

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

We sometimes denote this by $\exp(A)$. It can be shown that this is well-defined. Here, for a scalar $\lambda \in \mathbb{R}$, λA denotes the matrix A with every entry multiplied by λ.

For example,

$$\exp\left(\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}\right) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} + \begin{pmatrix} \frac{d_1^2}{2} & & & \\ & \frac{d_2^2}{2} & & \\ & & \ddots & \\ & & & \frac{d_n^2}{2} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{pmatrix}.$$

Theorem 10.1 (Fundamental theorem of linear systems)
The unique solution to (1) is

$$\mathbf{x} = e^{At}\mathbf{x}_0.$$

So, the problem is to compute e^{At}. To do this we use the Jordan canonical form of A over $\mathbb{R}$. Let P be an invertible $n \times n$ matrix such that

$$PAP^{-1} = J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_c \end{pmatrix},$$

where each J_i is a generalized Jordan block.

For simplicity, we shall assume that each J_i is a Jordan block, i.e. of the form

$$\begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

We substitute $\mathbf{y} = P\mathbf{x}$. Then:

$$\dot{\mathbf{y}} = P\dot{\mathbf{x}} = PA\mathbf{x} = PAP^{-1}\mathbf{y} = J\mathbf{y}.$$

The solution to this equation is
$$\mathbf{y} = e^{Jt}\mathbf{y}_0,$$
where $\mathbf{y}_0 = P\mathbf{x}_0$. Then we get the solution:
$$\mathbf{x} = P^{-1}\mathbf{y} = P^{-1}e^{Jt}P\mathbf{x}_0. \tag{2}$$

We have
$$e^{Jt} = \exp\left(\begin{pmatrix} J_1 t & & & \\ & J_2 t & & \\ & & \ddots & \\ & & & J_c t \end{pmatrix}\right) = \begin{pmatrix} \exp(J_1 t) & & & \\ & \exp(J_2 t) & & \\ & & \ddots & \\ & & & \exp(J_c t) \end{pmatrix}$$
so we may consider a single block
$$J_i = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & 1 & \lambda \end{pmatrix} = \lambda I_n + X$$
where
$$X = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$
If this is a $d \times d$ matrix, then note that $X^d = 0$, i.e. X is a nilpotent matrix. Then we have:
$$\begin{aligned} e^{Jt} &= \exp((\lambda I_d + X)t) \\ &= \exp(\lambda t I_d)\exp(Xt) \\ &= \begin{pmatrix} \exp(\lambda t) & & & \\ & \exp(\lambda t) & & \\ & & \ddots & \\ & & & \exp(\lambda t) \end{pmatrix} \begin{pmatrix} 1 & & & & \\ t & 1 & & & \\ \frac{t^2}{2!} & t & \ddots & & \\ \vdots & \ddots & \ddots & 1 & \\ \frac{t^{d-1}}{(d-1)!} & \cdots & \frac{t^2}{2!} & t & 1 \end{pmatrix} \end{aligned}$$

This allows us to compute $e^{J_i t}$ for each i, and hence e^{Jt} and therefore the solution (2) $\mathbf{x} = P^{-1}e^{Jt}P\mathbf{x}_0$ for $\mathbf{x}$.

For more information on this, see e.g. Section 1.8 of the book below (or other books on differential equations). In particular, this considers the case of the other (i.e. degree 2) irreducible polynomials over $\mathbb{R}$.

Lawrence Perko, Differential equations and dynamical systems. Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.

R. J. Marsh, 17/11/14.