MA3201 Rings and Modules, 2014

Problem Sheet 4

To be discussed on Friday 17 October and Friday 24 October.

Problems from: Bhattacharya, P. B.; Jain, S. K.; Nagpaul, S. R. Basic abstract algebra. Second edition. Cambridge University Press, Cambridge, 1994.

Page	Problem number
381	1 (see remark below)
381	3
381	8 (challenge question)
381	9 (assume D is a field)
388	1 (Use the Wedderburn-Artin theorem)

Problem 1. Let R be a ring.

(a) Let Λ be a set and let

$$M = \{(r_i)_{i\in\Lambda} : r_i \in R, \text{ finitely many } r_i \neq 0\}.$$

Show that M is a free R-module with basis indexed by the elements of Λ.

(b) Let M be an R-module. Show that there is a free R-module F with a homomorphism $\varphi : F \to M$ which is onto. Deduce that M is a quotient of a free module. Show that if M is finitely generated, then F can also be taken to be finitely generated.

Problem 2. Let R be a left noetherian ring and suppose that M is a finitely generated R-module. Show that M is a noetherian R-module.

Problem 3. Let R be a left noetherian ring. Show that $M_n(R)$ is a noetherian left R-module, with $r(a_{ij}) = (ra_{ij})$ for $r \in R$ and $(a_{ij}) \in M_n(R)$. Use this to show that $M_n(R)$ is itself a left noetherian ring.

Problem 4. Let R be a ring and M a semisimple R-module. Suppose that N is a submodule of N. Prove that M/N is semisimple. Use Problem 1 to show that ${}_RR$ is semisimple if and only if every R-module is semisimple.

Problem 5. Let R be a ring and suppose that R_R is semisimple. Show that R_R is a finite direct sum of minimal left ideals. (*Hint*: use Corollary 4.10 from lectures and consider an expression for $1 \in R_R$).

Problem 6. Let $\mathbb{F}$ be a field and Q the quiver $1 \longrightarrow 2 \longrightarrow 3$. Find a nilpotent ideal I in the path algebra $\mathbb{F}Q$ such that ${}_RR$ is semisimple, where $R = \mathbb{F}Q/I$.

Problem 7. Prove that ${}_{\mathbb{Z}}\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Remark on p381, Q1: If R_1, R_2 are rings then we have defined their direct product

$$R_1 \times R_2 = \{(r_1, r_2) : r_1 \in R_1, r_2 \in R_2\},$$

with componentwise addition and multiplication. A similar construction defines the direct product of rings $R_1, R_2, \ldots, R_n$:

$$R_1 \times R_2 \times \cdots \times R_n = \{(r_i)_{1 \le i \le n} : r_i \in R_i\}.$$

If R_i, $i \in \Lambda$, are rings, where Λ is any set, then we have the direct product:

$$\prod_{i \in \Lambda} R_i = \{(r_i)_{i \in \Lambda} : r_i \in R_i, i \in \Lambda\},$$

with componentwise addition and multiplication. The direct *sum* $\bigoplus_{i \in \Lambda} R_i$ is the subset of $\prod_{i \in \Lambda} R_i$ consisting of the tuples $(r_i)_{i \in \Lambda}$ in which only finitely many of the r_i are non-zero. But note that, in general, this may not have an identity element if Λ is infinite. If Λ is finite, the direct sum and direct product coincide.

R. J. Marsh, 8/10/14.