MA3201 Rings and Modules, 2014

Problem Sheet 5

To be discussed on Friday 7 November.

Problems from old exams.

Exam	Problem number
2012	2, 4
2011	2
2009	4

Problem 1. Let R be a ring, $I \subseteq R$ an ideal.

(a) Let M an R-module with the property that $rm = 0$ for all $r \in I$ and $m \in M$. Show that setting $(r + I)m = rm$ makes M into an R/I module.

(b) Show further that if N is any R/I-module, it can be made into an R-module in a similar way, with the property that $rn = 0$ for all $r \in I$ and $n \in N$.

(c) Show that starting with an R-module M with the property in (a) and following the procedures as in (a) and then (b), we obtain the same R-module structure back. Show further that starting with an R/I-module N and following the procedures as in (b) and then (a), we obtain the same R/I-module structure back.

Problem 2. Let R be a ring. Show that $M_n(R^{\text{opp}}) \cong M_n(R)^{\text{opp}}$. Show further that if R, S are rings then $(R \times S)^{\text{opp}} \cong R^{\text{opp}} \times S^{\text{opp}}$.

Problem 3. Let R be a ring and let M, N be R-modules. Let $\varphi : M \to N$ be an R-isomorphism. Let $\varphi^* : \text{End}_R(M) \to \text{End}_R(N)$ be the map taking f to $\varphi f \varphi^{-1}$ for $f \in \text{End}_R(N)$. Prove that φ^* is a ring isomorphism.

Problem 4. Let R be a ring. Use the Wedderburn-Artin Theorem to show that ${}_RR$ is semisimple if and only if R_R is semisimple.

Problem 5. Let R be a finite ring such that $x^2 = x$ for all $x \in R$.

(a) Show that R is commutative. (Hint. Consider the elements $(x + y)^2$ and $x^2 = (-x)^2$ for $x, y \in R$).

(b) Show that R is semisimple artinian.

(c) (Challenge) Use the Wedderburn-Artin Theorem to show that R is isomorphic as a ring to a finite direct product of copies of the field $\mathbb{Z}_2$.

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