MA3201 Rings and Modules, 2014

Solution Sheet 4

To be discussed on Friday 17 October and Friday 24 October.

Problems from: Bhattacharya, P. B.; Jain, S. K.; Nagpaul, S. R. Basic abstract algebra. Second edition. Cambridge University Press, Cambridge, 1994.

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381	1
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381	8 (challenge question)
381	9 (assume D is a field)
388	1 (Use the Wedderburn-Artin theorem)

Page 381-2, Q1

We consider the case of modules first. Suppose R is a ring and $M = \oplus_{i=1}^n M_i$ is a direct sum of n noetherian submodules $M_1, M_2, \ldots, M_n$. Then M is noetherian. We prove this by induction on n. It is clear that the case $n = 1$ is true, so suppose the result is true for smaller n. Then $M_1 \oplus \cdots \oplus M_{n-1}$ is noetherian. It is easy to check that the map $\varphi : M \to M_n$ mapping an element $m \in M$ written $m = m_1 + \cdots + m_n$ with $m_i \in M_i$ to m_n is a ring homomorphism. Its image is M_n and its kernel is $M_1 \oplus \cdots \oplus M_{n-1}$, both of which are noetherian, by assumption and the inductive hypothesis respectively. By the Fundamental Theorem of Homomorphisms,

$$\frac{M_1 \oplus \cdots \oplus M_n}{M_1 \oplus \cdots \oplus M_{n-1}} \cong M_n.$$

Hence, by Theorem 4.18 in lectures, M itself is noetherian. The result follows by induction.

For the case of rings, it is clear that the case $n = 1$ is true. Suppose that the result is true for smaller n. Let $S = R_1 \oplus \cdots \oplus R_n$. For $1 \leq i \leq n$, let

$$R_i' = \{(r_1, \ldots, r_n) \in S \,:\, r_j = 0 \text{ for } j \neq i, r_i \in R_i\}.$$

Then it is easy to check that R_i' is an S-submodule of S. We can also make R_i into an S-module by setting $(r_1, \ldots, r_n)x = r_i x$ for all $x \in R_i$. The map $\varphi_i : R_i \to S$ taking r_i to $(0, \ldots, 0, r_i, 0, \ldots, 0)$ (with r_i in the ith position) is easily seen to be an S-isomorphism. The S-submodules of R_i coincide with the R_i-submodules of R_i as a left R_i-module. Since R_i is left noetherian, R_i is a noetherian S-module, hence so is R_i'. By the above, the direct sum $R_1' \oplus \cdots \oplus R_n' = S$ is a noetherian S-module as required.

Page 381-2, Q3

Let R be a principal left ideal ring, i.e. every left ideal of R is principal (see page 183). Then every left ideal of R is finitely generated (in fact, generated by a single element), so, by Corollary 4.17 from lectures, R is left noetherian.

Page 381-2, Q8

Let $a \in R$, $a \neq 0$. Consider the descending sequence of left ideals of R:

$$Ra \supseteq Ra^2 \supseteq \cdots .$$

Since R is artinian, there is some k such that $Ra^k = Ra^{k+1}$, so $a^k = ba^{k+1}$ for some $b \in R$. Hence $(1 - ba)a^k = 0$. Since $a \neq 0$ and R is an integral domain, $a^k \neq 0$, so $1 - ba = 0$ and $ba = 1$. Then $(ab)a = a(ba) = a1 = a$. So, since $a \neq 0$, $ab = 1$, and b is an inverse for a. Since $R \neq \{0\}$, it is a division ring.

Page 381-2, Q9

We assume that $D = \mathbb{F}$ is a field, so R is a vector space over $\mathbb{F}$. Any left ideal of R is closed under left multiplication by elements of $\mathbb{F}$, so it is a left $\mathbb{F}$-module. Hence it is an $\mathbb{F}$-subspace of R. Therefore any descending chain

$$I_1 \supseteq I_2 \supseteq \cdots$$

of left ideals of R is a descending chain of $\mathbb{F}$-subspaces of R. Since R is finite dimensional as a vector space over $\mathbb{F}$, there is some k such that $I_i = I_k$ for all $i \geq k$. Hence R is left artinian.

Page 388, Q1

Let R be a left artinian ring with no nonzero nilpotent ideals. By the Wedderburn-Artin theorem, R has the form $R \cong M_{n_1}(D_1) \times \cdots M_{n_k}(D_k)$ for division rings $D_1, \ldots, D_k$. We have seen from lectures that the only ideals of $M_n(D)$ for a division ring D are $\{0\}$ and $M_n(D)$ itself. Hence the ideal I is of the form $I_1 \times \cdots \times I_k$ where each I_i is either $\{0\}$ or $M_{n_i}(D_i)$. Therefore R/I is isomorphic to a direct product of matrix rings over division rings. By the Wedderburn-Artin theorem, R/I is artinian with no nonzero nilpotent ideals.

Problem 1(a)

For $i \in \Lambda$, let $b_i = (r_j)_{j\in\Lambda}$, where $r_j = 1$ if $j = i$ and $r_j = 0$ otherwise. We will show that the b_i form a basis of M. If $r = (r_i) \in M$, then $r = \sum_{i\in\Lambda, r_i\neq 0} r_i b_i$, so the b_i generate M. Suppose $r = \sum_{i\in\Lambda'} r_i b_i = 0$ for some finite subset Λ' of Λ. Then $r = (r'_i)$ where $r'_i = r_i$ if $i \in \Lambda'$ and $r'_i = 0$ otherwise. So $r_i = 0$ for all $i \in \Lambda'$. So the b_i are linearly independent and therefore form a basis. Hence M is a free R-module. It is often denoted $R^{(\Lambda)}$.

Problem 1(b) Let F be the free module $R^{(M)}$ given by part (a), taking $\Lambda = M$. Define a map $\varphi : F \to M$ taking $(r_m)_{m\in M}$ to $\sum_{m\in M} r_m m$. It is easy to check that φ is an R-homomorphism. Since $\varphi(b_m) = m$, it is onto. Applying the Fundamental theorem of module homomorphisms to φ, we see that M is isomorphic to a quotient of F.

If M is finitely generated, let $x_1, \ldots, x_k$ be a generating set and take $\Lambda = \{1, \ldots, k\}$, so $F = R^k$. Define $\varphi : F \to M$ by sending b_i to x_i. It is easy to check that φ is an R-homomorphism. Since $x_1, \ldots, x_k$ lie in the image, it is onto.

Problem 2

By Problem 1, M is isomorphic to a quotient of R^k for some k. Since R is noetherian, R^k is a left noetherian R-module (by p381, Q1). Hence M is a noetherian R-module (by Theorem 4.18 in lectures).

Problem 3

It is easy to check that defining $r(a_{ij}) = (ra_{ij})$ makes $M_n(R)$ into an R-module. Furthermore, it is finitely generated, by the elementary matrices E_{ij} for $1 \leq i, j \leq n$, so by Problem 2, $M_n(R)$ is a noetherian R-module. Any left ideal of $M_n(R)$ must be an R-submodule, since $r(a_{ij}) = D(a_{ij})$ where D is the diagonal matrix with r in each diagonal position and zeros elsewhere. Hence any increasing sequence

$$I_1 \subseteq I_2 \subseteq \cdots$$

of left ideals of $M_n(R)$ is a sequence of R-submodules of $M_n(R)$ and therefore there is some k such that $I_k = I_i$ for all $i \geq k$.

Problem 4

Write $M = \sum_{i \in \Lambda} M_i$, where each M_i is a simple submodule of M. By Theorem 4.8 in lectures, we can write

$$M = N \oplus \bigoplus_{i \in \Lambda'} M_i,$$

where Λ' is a subset of Λ. Then M/N is isomorphic to

$$\bigoplus_{i \in \Lambda'} M_i$$

and hence is semisimple.

If every R-module is semisimple, then ${}_RR$, which is an R-module, is semisimple. Conversely, suppose that ${}_RR$ is semisimple, and suppose that M is an R-module. By Problem 1(b), M is a quotient of a free module F. The module F is a direct sum of modules, each of which is isomorphic to ${}_RR$ (use Q3, p268). Since each of these modules is semisimple, so is F, and hence so is M, since it is a quotient of M, using the above.

Problem 5

Let R be a ring and suppose that ${}_RR$ is semisimple. Then R is a sum of minimal left ideals. By Corollary 4.10 in lectures, R is a direct sum of minimal left ideals, ${}_RR = \bigoplus_{i \in \Lambda} I_i$. We delete any zero terms in the direct sum (since they do not affect the sum). Write $1 = x_1 + \cdots + x_k$ where $x_t \in I_{i_t}$ for $i_1, \ldots, i_k \in \Lambda$. Then any element $r \in R$ can be written as

$$r = r1 = rx_1 + \cdots rx_k,$$

with $rx_t \in I_{i_t}$. If $i \in \Lambda \setminus \{i_1, \ldots, i_k\}$ then, since $I_i \neq 0$, $I_i \cap R \neq \{0\}$, so $I_i \cap \sum_{t=1}^{k} I_{i_t} \neq \{0\}$, contradicting Proposition 3.13 in lectures. So $\Lambda = \{i_1, \ldots, i_t\}$ and we see that R is a finite direct sum of minimal left ideals as required.

Problem 6

Let α be the arrow from 1 to 2 and β the arrow from 2 to 3. Then we have seen (Problem 3 on Problem Sheet 1) that the subspace spanned by $\alpha, \beta, \beta\alpha$ is an ideal I of $\mathbb{F}Q$. It is easy to check that any product of two of these elements is zero, so the same is true for any linear combination and we see that I is nilpotent. It is easy to check that $R = \mathbb{F}Q/I$ is isomorphic to $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$, which as a left $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$-module is the sum of the simple $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$- submodules $\{0\} \times \mathbb{F} \times \mathbb{F}$, $\mathbb{F} \times \{0\} \times \mathbb{F}$ and $\mathbb{F} \times \mathbb{F} \times \{0\}$ and hence is semisimple.

Problem 7

If $p, q \in \mathbb{Z}$ with $q \neq 0$, then $\mathbb{Z}\frac{p}{q} \neq \mathbb{Q}$ (since $\frac{1}{2q} \notin \mathbb{Z}\frac{p}{q}$). So any basis must have at least two nonzero elements, say p/q and r/s, with $p, q, r, s \in \mathbb{Z}$ and $q, s \neq 0$. But then $rq\frac{p}{q} - ps\frac{r}{s} = rp - pr = 0$, so p/q and r/s are not linearly independent.

R. J. Marsh, 5/10/14.