MA3201 Rings and Modules, 2014

Solution Sheet 5

To be discussed on Friday 31 October and Friday 7 November.

Problems from old exams.

Exam	Problem number
2012	2, 4
2011	2
2009	4

Note: some details in the answers have been omitted.

Exam 2012, Problem 2.

We check that $X - X' \in I$ for all $X, X' \in I$ and $AX \in I$ and $XA \in I$ for all $A \in R$ and $X \in I$, to see that I is an ideal of R. Details omitted.

We have:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so R is not commutative.

The ring R is an $\mathbb{R}$-algebra, with

$$\lambda \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} \lambda a & 0 \\ \lambda b & \lambda c \end{pmatrix}$$

for $\lambda \in \mathbb{R}$. A basis for R over $\mathbb{R}$ is given by the elementary matrices E_{11}, E_{21} and E_{22}, so it is finite-dimensional over $\mathbb{R}$. Hence it is artinian and noetherian.

If X is an arbitrary matrix in I, then $X^2 = 0$, so I is a nilpotent ideal, which is also nonzero. Hence R has a nonzero nilpotent ideal, so it is not semisimple.

Define $\varphi : R \to \mathbb{R} \times \mathbb{R}$, setting $\varphi(X) = (a, c)$ for $X = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in R$. Then it can be checked that φ is an onto ring homomorphism, with kernel I. Applying the fundamental theorem of homomorphisms, we have that $R/\ker\varphi \cong \operatorname{im}\varphi$. Hence $R/I \cong \mathbb{R} \times \mathbb{R}$. So we can answer the question about R/I by answering it for the isomorphic ring $S = \mathbb{R} \times \mathbb{R}$.

Since $\mathbb{R}$ is commutative, we have $(a, b)(c, d) = (ac, bd) = (ca, db) = (c, d)(a, b)$ for $a, b, c, d \in \mathbb{R}$. Hence R/I is commutative.

The ring $\mathbb{R} \times \mathbb{R}$ is also an $\mathbb{R}$-algebra, with $\lambda(a, b) = (\lambda a, \lambda b)$ for $\lambda \in \mathbb{R}$, with basis $\{(1, 0), (0, 1)\}$. So it is a finite-dimensional $\mathbb{R}$-algebra, and hence artinian and noetherian.

We can write $\mathbb{R} \times \mathbb{R}$ as a direct sum of $X_1 = (\mathbb{R}, 0)$ and $X_2 = (0, \mathbb{R})$. Each of these can be seen to be minimal left ideals of S, and hence simple submodules of ${}_S S$. It follows that S (and hence R/I) is semisimple.

Let

$$m_1 = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{pmatrix}.$$

Then it can be checked that m_1 is an ideal of R. Consider the map $\alpha : R \to \mathbb{R}$, defined by $\alpha(X) = c$ for $X = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in R$. Then α is a ring homomorphism with kernel m_1 and image $\mathbb{R}$. By the fundamental theorem of homomorphisms, $R/m_1 \cong \mathbb{R}$, which is a field. Hence m_1 is a maximal ideal of R. We set

$$m_2 = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix};$$

a similar argument shows that m_2 is a maximal ideal. It is easy to check that $m_1 \cap m_2 = I$.

Let m be a maximal ideal of R and suppose m contains a matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $a \neq 0$. Then, for any $\lambda, \mu \in \mathbb{R}$, m also contains the matrix

$$\begin{pmatrix} \lambda/a & 0 \\ \mu/a & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \mu & 0 \end{pmatrix}$$

and hence contains m_1, so it must be equal to m_1. Similarly, if m contains a matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $c \neq 0$ then, by multiplying this matrix on the right by an appropriate matrix, we see that m contains m_2 and so must be equal to m_2. The only possibility left is that $m \subseteq I$. But then $m \subseteq m_1$ and $m \neq m_1$, so we have a contradiction. It follows that the only maximal ideals of R are m_1 and m_2.

We claim that m_1 is a maximal left ideal of R. Let I be a left ideal of R containing m_1 but not equal to m_1. Then I contains a matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $c \neq 0$. Hence, for any $\lambda \in \mathbb{R}$, I contains the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda/c \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda b/c & \lambda \end{pmatrix}.$$

Since it contains m_1, it contains the matrix:

$$\begin{pmatrix} 0 & 0 \\ -\lambda b/c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\lambda b/c & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Hence, since it is closed under addition and contains m_1, it is equal to R. So m_1 is a maximal left ideal of R. A similar argument shows that m_2 is also a maximal left ideal of R.

Hence the only R-submodules of ${}_RR$ containing m_1 are m_1 and R, so, by the correspondence theorem, the only R-submodules of $S_1 = ({}_RR)/m_1$ are the zero submodule and S_1 itself, so it is a simple R-module. Similarly, $S_2 = {}_RR/m_2$ is a simple R-module.

Suppose that $f : S_1 \to S_2$ was an isomorphism. Let $\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} + m_2$ be the image of $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m_1$. Then

$$f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m_1\right)\right) = f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m_1\right) = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} + m_2.$$

The left hand side must also be equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + m_1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} + m_2\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + m_2.$$

So $y = 0$, but

$$f\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + m_1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + m_2,$$

so f is not a bijection, a contradiction.

Hence S_1 and S_2 are nonisomorphic simple R-modules as required.

Exam 2012, Problem 4.

For part (a), suppose that M is a noetherian R module. Assume, for a contradiction, that M contains a submodule N which is not finitely generated. Choose an element $m_1 \in N$. Then $Rm_1 \subseteq N$, but $Rm_1 \neq N$ since N is not finitely generated. Choose $m_2 \in N \setminus Rm_2$. Then we have that the submodule generated by m_1 and m_2 is contained in N, but we do not have equality since N is not finitely generated. Repeating this argument, we obtain an ascending sequence of distinct submodules of N (and hence of

M), contradicting the assumption that M is noetherian. Hence every submodule of N is finitely generated.

For part (b), suppose that M is a noetherian and artinian R-module. Let $M_0 = M$. Suppose that $M \neq \{0_M\}$. Then the set of submodules of M not equal to M is nonempty (it contains $\{0_M\}$). Since M is noetherian, this set contains a maximal element, M_1. Since M_1 is a maximal submodule of M, the only submodules of M containing M_1 are M_1 and M itself. By the correspondence theorem, the only submodules of M/M_1 are M_1/M_1 and M/M_1. Also, $M \neq M_1$, so M/M_1 is not a zero module, so it is a simple R-module.

If M_1 is not equal to zero, then the set of submodules of M_1 not equal to M_1 is nonempty (containing the zero submodule). Hence, using again the fact that M is noetherian, the set of submodules of M_1 not equal to M_1 has a maximal element, M_2. As before, M_1/M_2 is a simple R-module.

We repeat the argument. Suppose that $M_n \neq \{0\}$ for all $n \geq 0$. Then we obtain a decreasing sequence

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of distinct submodules of M, contradicting the fact that M is artinian. Hence, there must be some n such that $M_n = 0$, and we obtain the required sequence

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = \{0\},$$

such that M_i/M_{i+1} is simple for each i.

Consider the ring $R = \mathbb{Z}$ and the $\mathbb{Z}$-module ${}_{\mathbb{Z}}\mathbb{Z}$. Since $\mathbb{Z}$ is a PID, every left ideal of $\mathbb{Z}$ is finitely generated, so ${}_{\mathbb{Z}}\mathbb{Z}$ is a noetherian $\mathbb{Z}$-module. We have the strictly decreasing sequence

$$(2) \supseteq (2^2) \supseteq (2^3) \supseteq \cdots$$

of submodules of ${}_{\mathbb{Z}}\mathbb{Z}$, so it is not artinian.

Suppose that

$${}_{\mathbb{Z}}\mathbb{Z} = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = \{0\}$$

was a finite decreasing sequence of submodules of ${}_{\mathbb{Z}}\mathbb{Z}$. Then, since $\mathbb{Z}$ is a PID, there is $a \in \mathbb{Z}$ such that $M_{n-1} = (a)$. But then $(2a)$ is a submodule of M_{n-1} which is not equal to M_{n-1} or $\{0\}$, so M_{n-1} (and hence $M_{n-1}/M_n = M_{n-1}/\{0\}$) is not a simple $\mathbb{Z}$-module. So there cannot be such a finite sequence in which all the quotients M_i/M_{i+1} are simple $\mathbb{Z}$-modules.

Exam 2011, Problem 2.

Part (a): to check that R is a subring of $M_3(\mathbb{F})$ we show that $AB \in R$ and $A - B \in R$ for all $A, B \in R$, and also that the identity matrix lies in R. Details omitted.

Part (b): to check that I_1 and I_2 are ideals of R, we check that $X - Y \in I$ for all $X, Y \in I$, and $AX, XA \in I$ for all $A \in R$ and $X \in I$. Details omitted.

Part (c): To answer this, could choose three of the following:

(a) R is left artinian with no non-zero nilpotent ideals.
(b) R is left artinian with no non-zero nil ideals.
(c) ${}_R R$ is a finite direct sum of minimal left ideals.
(d) Each left ideal of R is of the form Re for some idempotent e.

Part (d): Define $\varphi_1 : R \to R_1 = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$ by

$$\varphi_1\left(\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}\right) = \begin{pmatrix} a & c \\ 0 & e \end{pmatrix}.$$

Then it can be checked that φ_1 is a ring homomorphism with kernel I_1 and image R_1. So, by the Fundemental Theorem of Homomorphisms, $R/I_1 \cong R_1$. The set

$$I = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \right\}$$

can be checked to be an ideal of R_1. Since the square of any matrix in I is zero, it is a nilpotent ideal. So I contains a nonzero nilpotent ideal, and hence, by the Wedderburn-Artin theorem, R_1 is not semisimple. It follows that R/I_1 is not semisimple.

Define $\varphi_2 : R \to R_2 = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$, sending a matrix to the three entries on its diagonal. Then it can be checked that φ_2 is a ring homomorphism with kernel I_2 and image R_2. So, by the Fundamental Theorem of Homomorphisms, $R/I_2 \cong R_2$. Let

$$\begin{aligned} X_1 &= \mathbb{F} \times \{0\} \times \{0\}; \\ X_2 &= \{0\} \times \mathbb{F} \times \{0\}; \\ X_3 &= \{0\} \times \{0\} \times \mathbb{F}. \end{aligned}$$

Then it can be checked that each X_i is a minimal left ideal of R_2 and therefore a simple submodule of ${}_{R_2}R_2$. Furthermore, ${}_{R_2}R_2 = X_1 + X_2 + X_3$, so it is a sum of simple R_2-submodules. Hence ${}_{R_2}R_2$ is semisimple, so R_2 is a semisimple ring (and therefore so is R/I_2) as required.

Exam 2009, Problem 4.

Let A be a left ideal in a ring R, and assume that $A = Aa$ for some $a \in A$, $a \neq 0$.

Part (i). Since $a \in A$, $a \in Aa$ so there is some $e \in A$ such that $a = ea$. Since $a \neq 0$, $ea \neq 0$. Furthermore, $(e^2 - e)a = eea - ea = ea - ea = 0$.

Part (ii). Let

$$B = \{x \in A : xa = 0\}.$$

Let $x, y \in B$. Then $xa = 0$ and $ya = 0$. So $(x - y)a = xa - ya = 0 - 0 = 0$. Hence $x - y \in B$. Let $r \in R$ and $x \in B$. Then $xa = 0$. So $(rx)a = r(xa) = r0 = 0$. Hence $rx \in B$. Therefore B is a left ideal of R.

Part (iii). We assume that A is a minimal left ideal. By its definition, B is contained in A, and we have seen that B is a left ideal of R. We have that $e \in A$ and $ea \neq 0$. Hence $e \notin B$. So $B \neq A$. Since A is a minimal left ideal, we must have $B = 0$. Since $e \in A$, $e^2 - e \in A$. By part (i), $(e^2 - e)a = 0$, so $e^2 - e \in B$. Hence $e^2 - e = 0$, so $e^2 = e$ and e is an idempotent element.

Problem 1 from Problem Sheet 5.

Part (a): Let M be an R-module with the property that $rm = 0$ for all $r \in I$. Then set $(r + I)m = rm$ for all $r \in R$. We check this is well-defined: if $r + I = r' + I$ for $r, r' \in R$, then $r - r' \in I$ so $(r - r')m = 0$, so $rm = r'm$. The module axioms for M over the ring R/I follow from the fact that the corresponding axioms hold for M over the ring R.

Part (b): Let N be an R/I module. Set $rn = (r + I)n$ for any $r \in R$ and $n \in N$. The module axioms for N over the ring R follow from the fact that the corresponding axioms hold for N over the ring R/I. Furthermore, if $r \in I$, then $rn = (r + I)n = (0 + I)n = 0$.

Part (c): Let M be an R-module satisfying the condition in (a). Then by (a) we have an R/I-module structure on M given by $(r + I)m = rm$ for all $r \in R$ and $m \in M$. The construction in B gives a new R-module structure on M given by $r \cdot m = (r + I)m$. Hence $r \cdot m = rm$, so the two module structures are the same. The second part is similar.

Problem 2 from Problem Sheet 5.

Let $a*b = ba$ denote the product of elements $a, b \in R^{\text{opp}}$. Similarly, for two matrices A, B in $M_n(R)^{\text{opp}}$ we shall denote their product by $A * B = BA$. Define a map $\varphi : M_n(R^{opp}) \to M_n(R)^{\text{opp}}$ taking a matrix $A \in M_n(R^{\text{opp}})$ to its transpose (defined by $A^T = (A_{ji})_{1\le i,j\le n}$).

Let $A, B \in M_n(R^{\text{opp}})$. Then the i, j-entry of $\varphi(AB) = (AB)^T$ is given by

$$\begin{aligned}((AB)^T)_{ij} &= (AB)_{ji}\\ &= \sum_{k=1}^{n} A_{jk} * B_{ki}\\ &= \sum_{k=1}^{n} B_{ki}A_{jk}\\ &= \sum_{k=1}^{n} (B^T)_{ik}(A^T)_{kj}\\ &= (B^T A^T)_{ij}.\end{aligned}$$

Hence $\varphi(AB) = \varphi(B)\varphi(A) = \varphi(A) * \varphi(B)$.

It is easy to check that $\varphi(A + B) = \varphi(A) + \varphi(B)$ and that $\varphi(1_{M_n(R^{\text{opp}})}) = 1_{M_n(R)^{\text{opp}}}$, so φ is a ring homomorphism. The map $\psi : M_n(R)^{\text{opp}} \to M_n(R^{\text{opp}})$ also sending a matrix to its transpose can be seen to be an inverse of φ, so φ is a ring isomorphism.

For the second part, note that $(R \times S)^{\text{opp}}$ and $R^{\text{opp}} \times S^{\text{opp}}$ are both equal, as sets, to $R \times S$. If (r, s), (r', s') are elements of $R \times S$ then their product in $(R \times S)^{\text{opp}}$ is the product $(r', s')(r, s)$ in $R \times S$, which is equal to $(r'r, s's)$. Their product in $R^{\text{opp}} \times S^{\text{opp}}$ is $(r * r', s * s') = (r'r, s's)$, which is the same. It is easy to check that addition is the same for the two structures and that the identity element is the same. So the map taking (r, s) in $(R \times S)^{\text{opp}}$ to (r, s) in $R^{\text{opp}} \times S^{\text{opp}}$ is a ring isomorphism.

Problem 3 from Problem Sheet 5.

Firstly, note that since φ and f are both R-endomorphisms, so is the composition $\varphi f \varphi^{-1} : N \to N$. Let $f, g \in \mathrm{End}_R(M)$ and $n \in N$. Then

$$\begin{aligned}\varphi^*(f + g)(n) &= (\varphi(f + g)\varphi^{-1})(n)\\ &= \varphi((f + g)(\varphi^{-1}(n)))\\ &= \varphi(f(\varphi^{-1}(n)) + g(\varphi^{-1}(n)))\\ &= \varphi(f(\varphi^{-1}(n))) + \varphi(g(\varphi^{-1}(n)))\\ &= (\varphi f \varphi^{-1})(n) + (\varphi g \varphi^{-1})(n)\\ &= \varphi^*(f)(n) + \varphi^*(g)(n).\end{aligned}$$

Hence $\varphi^*(f + g) = \varphi^*(f) + \varphi^*(g)$.

Furthemore,

$$\begin{aligned}\varphi^*(fg)(n) &= (\varphi f g \varphi^{-1})(n)\\ &= (\varphi f \varphi^{-1}\varphi g \varphi^{-1})(n)\\ &= (\varphi^*(f)\varphi^*(g))(n).\end{aligned}$$

Hence $\varphi^*(fg) = \varphi^*(f)\varphi^*(g)$.

And

$$\begin{aligned}\varphi^*(1_M)(n) &= (\varphi 1_M \varphi^{-1})(n)\\ &= \varphi\varphi^{-1}(n) = n = 1_N(n).\end{aligned}$$

So $\varphi^*(1_M) = 1_N$, and we see that φ^* is a ring homomorphism. Let $\psi : \mathrm{End}_R(N) \to \mathrm{End}_R(M)$ be the map taking $g \in \mathrm{End}_R(N)$ to $\varphi^{-1} g \varphi$. Then, for $f \in \mathrm{End}_R(M)$, we have $\psi\varphi^*(f) = \varphi^{-1}(\varphi f \varphi^{-1})\varphi = f$ and similarly $\varphi^*\psi(g) = g$ for $g \in \mathrm{End}_R(N)$. Hence φ is a ring isomorphism.

Problem 4 from Problem Sheet 5.

Suppose that ${}_R R$ is semisimple. By Problem 5 on Problem Sheet 4, R is a direct sum of finitely many minimal left ideals. By the Artin-Wedderburn theorem, R is a direct product of finitely many matrix rings over division rings. Hence R^{opp} is also, by Problem 2 on this problem sheet. Applying the Artin-Wedderburn Theorem to R^{opp}, we see that R^{opp} is a finite direct sum of finitely many minimal left ideals, so R is a finite direct sum of finitely many minimal right ideals. Hence R_R is semisimple. The converse is proved similarly.

Problem 5 from Problem Sheet 5.

Part (a). Let $x, y \in R$. Then

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y,$$

so $xy + yx = 0$. And

$$x = x^2 = (-x)^2 = -x,$$

so (applying this second equality to yx), $xy = -yx = yx$. Hence R is commutative.

Part (b). Suppose that I is a nilpotent ideal of R, so $I^n = \{0\}$ for some $n \in \mathbb{N}$. Let $x \in I$. Then $x^n \in I^n$ so $x^n = 0$. But $x^n = x$ (repeatedly applying $x^2 = x$). So $x = 0$. Hence $I = 0$ so R has no nonzero nilpotent ideals.

If

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

is a descending sequence of left ideals in R then (noting R is finite):

$$|R| \geq |M_0| \geq |M_1| \geq |M_2| \geq \cdots .$$

There are only finitely i for which $|M_i| > |M_{i+1}|$ (since R is finite), so there must be some k for which $|M_i| = |M_k|$ for all $i \geq k$. But since each M_i is finite and $M_i \subseteq M_k$ for all $i \geq k$, we have $M_i = M_k$ for all $i \geq k$, so R is artinian.

So R is artinian and has no nilpotent ideals. Hence R is semisimple by the Wedderburn-Artin theorem.

Part (c). By the Wedderburn-Artin theorem, R must be a direct product of finitely many matrix rings over division rings.

$$R = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Fix p with $1 \leq p \leq k$. Then, for all $x \in M_{n_p}(D_p)$ we must have $x^2 = x$. Since D_p is nonzero it has a nonzero element a. If $i \neq j$ then in $M_{n_p}(D_p)$ we have $(I_{n_p} + aE_{ij})^2 = I_{n_p} + 2aE_{ij} = I_n + aE_{ij}$, so $a = 0$, a contradiction. It follows that $n_p = 1$ (so we cannot have $i \neq j$ in the above). Hence

$$R \cong D_1 \times \cdots D_k.$$

If $x \in D_p$, $x^2 = x$, so $x(x - 1_{D_p}) = 0_{D_p}$. Since D_p is an integral domain, $x = 0_{D_p}$ or $x = 1_{D_p}$, so $D_p = \{0_{D_p}, 1_{D_p}\}$. These elements are distinct as D_p is not the zero ring (as it is a division ring). If $1_{D_p} + 1_{D_p} = 1_{D_p}$ then we have $1_{D_p} = 0_{D_p}$, a contradiction. So $1_{D_p} + 1_{D_p} = 0_{D_p}$. It follows that $D_p \cong \mathbb{Z}_2$. Hence R is isomorphic to a direct product of copies of $\mathbb{Z}_2$. (We note that such a direct product satisfies the original assumptions).

R. J. Marsh, 30/10/14.