NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **MA3201 Rings and Modules**

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Problem 1

a) Find the Smith normal form of the matrix

$$\begin{pmatrix} 2-X & 1 & 2 \\ 0 & 1-X & 2 \\ 1 & 0 & 1-X \end{pmatrix}$$

over $\mathbb{Z}_5[X]$.

b) Find the rational canonical form of the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{Z}_5$.

c) Let $M_3(\mathbb{Z}_5)$ be the ring of 3×3 matrices over $\mathbb{Z}_5$ and define $\Phi_A : \mathbb{Z}_5[X] \to M_3(\mathbb{Z}_5)$ by letting $\Phi_A(P) = P(A)$ for each polynomial P in $\mathbb{Z}_5[X]$. The Image of Φ_A is then the subring of $M_3(\mathbb{Z}_5)$ generated by the matrix A. Prove that this subring generated by the matrix A is not a field.

Problem 2 Let $\Lambda = \{\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_6\} \subset M_3(\mathbb{Z}_6)$, the ring of 3×3 matrices over $\mathbb{Z}_6$.

a) Prove that Λ is a commutative subring of $M_3(\mathbb{Z}_6)$, the ring of 3×3 matrices over $\mathbb{Z}_6$.

b) Define $\Psi : \Lambda \to \mathbb{Z}_6$ by $\Psi(\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}) = a+b+c$. Prove that Ψ is a surjective ring homomorphism and find a set of generators for the kernel of Ψ.

c) How many maximal ideals in Λ contain the kernel of Ψ? You have to give an argument for your answer.

d) Is Λ a semisimpel ring? You have to give an argument for your answer.

Problem 3 Let R be a ring and A and C left submodules of the left R-module B, i.e. $A \subseteq B$ and $C \subseteq B$.

a) Prove that the submodule $A + C$ of B is finitely generated if A and C are finitely generated.

b) Prove that if A, C and $B/(A + C)$ are all finitely generated, the B is also finitely generated.

c) Prove that B is artinian if and only if A, C and $B/(A + C)$ are all artinian.