EXERCISES

26.08. - 07.09.

Exercise 1. Let $Q = \left[1 \longrightarrow 2\right]$. Consider the representations $V = \left[k \xleftarrow{\mathrm{id}} k\right]$ and $W = \left[k \xleftarrow{0} k\right]$. Let $\varphi\colon V \longrightarrow W$ be given by $\varphi(1) = 0$ and $\varphi(2) = \mathrm{id}$.

Calculate kernel, image, and cokernel of φ.

Exercise 2. Show that in $\mathrm{rep}(Q, R)$ a homomorphism is

- a monomorphism iff it is injective at every vertex,
- an epimorphism iff it is surjective at every vertex,
- an isomorphism iff it is bijective at every vertex.

09.09. - 14.09.

Exercise 3. Let $\mathscr{C} = \mathrm{rep}(1 \longrightarrow 2)$ and $\mathscr{D} = \mathrm{rep}(1 \longrightarrow 2 \longrightarrow 3)$. Consider the functors

$$\begin{aligned}
&\mathtt{F}\colon \mathscr{D} \longrightarrow \mathscr{C}\colon (V_1 \longleftarrow V_2 \longleftarrow V_3) \longmapsto (V_1 \longleftarrow V_2),\\
&\mathtt{G}\colon \mathscr{C} \longrightarrow \mathscr{D}\colon (V_1 \longleftarrow V_2) \longmapsto (V_1 \longleftarrow V_2 \longleftarrow 0), \text{ and}\\
&\mathtt{H}\colon \mathscr{C} \longrightarrow \mathscr{D}\colon (V_1 \longleftarrow V_2) \longmapsto (V_1 \longleftarrow V_2 \xleftarrow{\mathrm{id}} V_2).
\end{aligned}$$

Show that $(\mathtt{G}, \mathtt{F})$ and $(\mathtt{F}, \mathtt{H})$ are adjoint pairs.

Exercise 4. Show the *Yoneda Lemma*:

Let $\mathscr{C}$ be a category, $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathbf{Set}$ a covariant functor, and C an object in $\mathscr{C}$. There is a bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(C, -), \mathtt{F}) \longrightarrow \mathtt{F}(C).$$

(Here the left side denotes the set of natural transformations from $\mathrm{Hom}_{\mathscr{C}}(C, -)$ to $\mathtt{F}$.)

15.09. - 21.09.

Exercise 5. Let $\mathscr{C}$ be a category. Show that

- $f \circ g$ mono $\Longrightarrow$ g mono,
- f and g mono $\Longrightarrow$ $f \circ g$ mono,
- f mono and split-epi $\Longrightarrow$ f iso. Find an example of a morphism that is mono and epi, but not iso.

Exercise 6. Find a quiver Q and a representation V such that

$$\mathrm{Hom}_{\mathrm{rep}\, Q}(V, -)\colon \ \mathrm{rep}\, Q \longrightarrow \mathrm{mod}\, k$$

does not commute with colimits. (*Hint*: cokernels)

Exercise 7. Show that in the definition of additive category, (3) can be replaced by "any two objects have a product".

23.09. - 28.09.

Exercise 8. Prove

- the 3×3 Lemma
- the 5 Lemma
- the Snake Lemma

for abelian categories. If this turns out to be difficult, restrict to the case of modules over a ring.

30.09. - 26.10.

Exercise 9. Let $L \in \operatorname{Mod} R$, M an R-S-bimodule, and $N \in \operatorname{Mod} S^{\mathrm{op}}$.
Show

$$(L \bigotimes_R M) \bigotimes_S N \cong L \bigotimes_R (M \bigotimes_S N).$$

Hint On elementary tensors, the map from left to right should send $(l \otimes m) \otimes n$ to $l \otimes (m \otimes n)$. The main issue is to show that this gives a well-defined map.

Exercise 10. We call two short exact sequences starting in A and ending in B equivalent if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & E_2 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Show:

- This defines an equivalence relation.
- In **Ab**, find all equivalence classes of short exact sequences
  - $0 \longrightarrow \mathbb{Z}/(2) \longrightarrow ? \longrightarrow \mathbb{Z}/(3) \longrightarrow 0$,
  - $0 \longrightarrow \mathbb{Z}/(2) \longrightarrow ? \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$.
- In $\operatorname{rep}\begin{pmatrix} 1 \\ \uparrow \\ 2 \end{pmatrix}$, find all equivalence classes of short exact sequences
  - $0 \longrightarrow \begin{smallmatrix} k \\ \downarrow \\ 0 \end{smallmatrix} \longrightarrow ? \longrightarrow \begin{smallmatrix} 0 \\ \downarrow \\ k \end{smallmatrix} \longrightarrow 0$,
  - $0 \longrightarrow \begin{smallmatrix} 0 \\ \downarrow \\ k \end{smallmatrix} \longrightarrow ? \longrightarrow \begin{smallmatrix} k \\ \downarrow \\ 0 \end{smallmatrix} \longrightarrow 0$.

08.10. - 26.10.

Exercise 11. Show the *Horseshoe Lemma*:
Given the diagram

$$\begin{tikzcd}
 & P \arrow[d, two heads] & & Q \arrow[d, two heads] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

in an abelian category, where the horizontal sequence is short exact, both vertical maps are epimorphisms, and Q is projective.

Show that this can be completed to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where the upper sequence is also exact, and the middle vertical arrow is also an epimorphism.

Exercise 12. Let Q be a finite quiver without oriented cycles. (That is there is no path from one vertex back to the same vertex, except for the lazy path at that vertex.)

(1) For $i \in Q_0$, let S_i be the representation given by
- for a vertex j of Q: $S_i(j) = \begin{cases} k & \text{for } j = i \\ 0 & \text{otherwise} \end{cases}$;
- for an arrow α of Q: $S_i(\alpha) = 0$.

Let R be any non-zero finite dimensional representation of Q. Show that there is $i \in Q_0$ such that there is a monomorphism $S_i \rightarrowtail R$.

(2) For $i \in Q_0$, let P_i be the representation given by
- for a vertex j of Q: $P_i(j) = k^{\{\text{paths from } j \text{ to } i\}}$ (the vector space of formal linear combinations of paths from j to i);
- for an arrow $\alpha \colon j \longrightarrow j'$:
$$P_i(\alpha) \colon k^{\{\text{paths from } j' \text{ to } i\}} \longrightarrow k^{\{\text{paths from } j \text{ to } i\}}$$
is given by composing paths with α.

Show that P_i is a projective object in $\operatorname{rep}(Q)$.

(3) For $i \in Q_0$, let I_i be the representation given by
- for a vertex j of Q: $I_i(j) = k^{\{\text{paths from } i \text{ to } j\}}$;
- for an arrow $\alpha \colon j \longrightarrow j'$:
$$I_i(\alpha) \colon k^{\{\text{paths from } i \text{ to } j'\}} \longrightarrow k^{\{\text{paths from } i \text{ to } j\}}$$
is given by sending a path $\alpha_n \cdots \alpha_1$ to $\begin{cases} \alpha_{n-1} \cdots \alpha_1 & \text{if } \alpha_n = \alpha \\ 0 & \text{otherwise} \end{cases}$.

Show that I_i is an injective object in $\operatorname{rep}(Q)$.

(4) Let $R \in \operatorname{rep}(Q)$. Show that there is a projective object $P \in \operatorname{rep}(Q)$ such that there is an epimorphism $P \twoheadrightarrow R$. In this situation we say that the category $\operatorname{rep}(Q)$ has *enough projectives*.

(5) Let $R \in \operatorname{rep}(Q)$. Show that there is an injective object $I \in \operatorname{rep}(Q)$ such that there is a monomorphism $R \rightarrowtail I$. In this situation we say that the category $\operatorname{rep}(Q)$ has *enough injectives*.

(6) Let $Q = \left[\begin{tikzcd}[sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd} \right]$. Find the representations S_1, S_2, S_3, P_1, P_2, P_3, I_1, I_2, and I_3. (That is, write them down as a picture containing a vector space in each vertex and a matrix over each arrow.)

28.10. - 02.11.

Exercise 13. Find projective resolutions for all the representations S_i
- of the quiver $Q = [1 \longrightarrow 2]$;
- of the quiver $Q = \left[\begin{tikzcd}[sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd} \right]$;

- of the quiver with relations $(Q,R) = (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \beta\alpha)$.

04.11. - 09.11.

Exercise 14. Consider the quiver with relations (Q,R) as in the third point of Exercise 13 above.

- Calculate $\operatorname{Ext}^n_{\operatorname{rep}(Q,R)}(S_i,S_j)$ for all $i,j \in \{1,2,3\}$ and $n \in \mathbb{N}$.
- Show that $\operatorname{Ext}^n_{\operatorname{rep}(Q,R)}(M,N) = 0$ for all representations M and N and all $n \geqslant 3$.

11.11. - 16.11.

Exercise 15. Let A be a finitely generated abelian group. Compute $\operatorname{Hom}_{\mathbf{Ab}}(A,\mathbb{Z})$ and $\operatorname{Ext}^1_{\mathbf{Ab}}(A,\mathbb{Z})$.

Exercise 16. Let $\mathscr{A}$ be a hereditary abelian category. Assume we are given the solid part of the following commutative diagram with exact rows and columns. Show that it is possible to find the dashed part.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail, dashed] & C \arrow[d, tail] \\
A \arrow[r, tail, dashed] & H \arrow[r, two heads, dashed] \arrow[d, two heads, dashed] & D \arrow[d, two heads] \\
& E \arrow[r, equal] & E
\end{tikzcd}$$

Exercise 17. Let (Q,R) be the quiver $Q = [1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n]$ with relations $\alpha_2\alpha_1, \alpha_3\alpha_2, \ldots, \alpha_{n-1}\alpha_{n-2}$.

Compute $\operatorname{Ext}^m_{\operatorname{rep}(Q,R)}(S_i,S_j)$ for all $i,j \in \{1,\ldots,n\}, m \in \mathbb{N}$.

17.11. - 23.11.

Exercise 18. The aim of this exercise is to determine the right derived functors of Ker.

Let $\mathscr{A}$ be an abelian category with enough injectives. We also consider the abelian category $\mathscr{M} = \operatorname{rep}_{\mathscr{A}}(1 \longleftarrow 2)$ – that is the categories whose objects are morphisms in $\mathscr{A}$, and whose morphisms are commutative squares.

Now Ker defines a left exact functor $\mathscr{M} \longrightarrow \mathscr{A}$. To determine the right derived functors, proceed as follows:

(1) Show that for I injective in $\mathscr{A}$, the representations $I \longrightarrow 0$ and $I \xrightarrow{\operatorname{id}_I} I$ are injective in $\mathscr{M}$.

(2) Use (1) to find an injective resolution of $X \oplus Y \overset{(0\ 1)}{\twoheadrightarrow} Y$ in $\mathscr{M}$. (In terms of the injective resolutions of X and Y.)

(3) Show that $\mathrm{R}^n \operatorname{Ker}(X \oplus Y \overset{(0\ 1)}{\twoheadrightarrow} Y) = 0$ whenever $n \geqslant 1$.

(4) For an arbitrary representation $X \xrightarrow{f} Y$, use a short exact sequence

$$0 \longrightarrow \begin{matrix} X \\ \big\downarrow f \\ Y \end{matrix} \longrightarrow \begin{matrix} X \oplus Y \\ \big\downarrow (0\ 1) \\ Y \end{matrix} \longrightarrow \begin{matrix} Y \\ \big\downarrow 0 \\ 0 \end{matrix} \longrightarrow 0$$

to determine $\mathrm{R}^n \operatorname{Ker}(X \xrightarrow{f} Y) = 0$.