HOMOLOGICAL ALGEBRA

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I. **Quiver representation.**
Definitions.

II. **Categories, functors, natural transformations.**
Definitions. Examples, in particular Hom-functors.

III. **Equivalences, adjoints, limits.**
Definitions. Unit and counit of an adjunction. In particular (co)products, (co)equalizers, (co)kernels as examples of (co)limits.

Thm. *Right adjoints commute with limits.*
Left adjoints commute with colimits.

IV. **Additive and abelian categories.**
Definitions. Examples and non-examples. Basic properties.

V. **Complexes, homology, exactness.**
Definitions. Reformulation of mono/epi in terms of exact sequences. Definition of exact functor, left/right exact functor.
Hom **is left exact.**
Projective and injective objects.
Characterization of pullback/pushout. 3×3 **lemma.** 5 **lemma. Snake lemma.**

VI. **Tensor products.**
Definition. Existence. Tensor products as functors. Right exactness.

Thm (Hom-tensor adjunction)**.** *For L an R-module, M an R-S-bimodule, N an S^{op}-module*

$$\mathrm{Hom}_S(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_S(M, N)).$$

VII. **Homology and homotopy.**
Definitions.
Long exact sequence of homology.
Homotopy category. Homology well-defined on homotopy category. Projective and injective resolutions, these define functors $\mathscr{A} \longrightarrow \mathsf{K}(\mathscr{A})$.

VIII. **Derived functors.**
Definition.
Long exact sequence. Ext**.** Tor**.**
Syzygies. Dimension shift.

IX. Ext **and extensions.**
Definition of Yoneda-Ext. Baer sum.

Thm. *If $\mathscr{A}$ has enough injectives then*

$$\mathrm{YExt}^n_{\mathscr{A}}(X, Y) = \mathrm{Ext}^n_{\mathscr{A}}(X, -)(Y).$$

If $\mathscr{A}$ has enough projectives then

$$\mathrm{YExt}^n_{\mathscr{A}}(X, Y) = \mathrm{Ext}^n_{\mathscr{A}}(-, Y)(X).$$

X. **(Small) global dimension.**
Definition of gl.dim, semi-simple, hereditary. Lengths of projective/injective resolutions. $\mathrm{mod}\,\mathbb{Z}$ is hereditary.

XI. **Cones, quasi-isomorphisms, balancing** Tor **and** Ext**.**
Definitions. f quasi-iso $\iff$ $\mathrm{Cone}(f)$ exact. Double complex. Total complex.

Thm. *Let M and N be a left and right R-module. Then*

$$\mathrm{Tor}^R_n(M, -)(N) = \mathrm{Tor}^R_n(-, N)(M).$$

Thm. *Assume $\mathscr{A}$ has enough projectives and enough injectives. Then*

$$\mathrm{Ext}^n_{\mathscr{A}}(X, -)(Y) = \mathrm{Ext}^n_{\mathscr{A}}(-, Y)(X).$$

XII. **Triangulated categories.**
Definitions. Basic properties. $\mathsf{K}(\mathscr{A})$ is triangulated.

Thm. *If $\mathscr{A}$ has enough projectives then*

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{\mathsf{K}(\mathscr{A})}(\mathsf{p}X, \mathsf{p}Y[n]).$$

If $\mathscr{A}$ has enough injectives then

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{\mathsf{K}(\mathscr{A})}(\mathsf{i}X, \mathsf{i}Y[n]).$$

XIII. **Derived categories.**
Definition. Roofs and their composition.

Thm.

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(X, Y[n]).$$