EXERCISES

29.08. - 03.09.

Exercise 1. Let X be the poset $\{a \geqslant 0 \leqslant b\}$ (with a and b uncomparable).

- Determine all objects $F \in \operatorname{presh}_{\mathbf{Set}} X$ such that $F(i) \in \{\emptyset, \{\star\}\}\ \forall i \in X$.
- Which of the presheaves determined above are isomorphic to a presheaf of the form $\operatorname{Hom}_{\mathscr{C}_X}(-, i)$ for some $i \in X$?
- Determine all objects $F \in \operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$ (where $\mathbb{F}$ is a field), such that $F(i) = \mathbb{F}$ for all $i \in X$.
- Which of the presheaves determined in the third part are isomorphic?

Exercise 2. Let $X = \{1 \leqslant 2\}$ and $Y = \{1\}$. Let $\mathscr{C}$ be any category.

- Convince yourself that inclusion of Y into X induces a functor
$$\operatorname{presh}_{\mathscr{C}} X \longrightarrow \operatorname{presh}_{\mathscr{C}} Y : F \longmapsto F \circ \operatorname{incl}.$$
- Find a right adjoint to the functor above.
- For $\mathscr{C} = \mathbf{Set}$ or $\mathscr{C} = \mathbf{Ab}$, find a left adjoint to the functor above.

05.09. - 10.09.

Exercise 3. Let X be any poset, and F a $\mathbf{Set}$-valued presheaf on X. Show

- that the limit $\varprojlim F$ exists;
- that the colimit $\varinjlim F$ exists.

Hint: Construct them explicitely, starting with product and coproduct, respectively.

Exercise 4. In the category $\mathbf{Ab}$

- Show that the pullback of
$$\begin{tikzcd} & L \arrow[d, "\alpha"] \\ M \arrow[r, "\beta"] & N \end{tikzcd}$$
is given by
$$L \prod_N M = \{(l, m) \in L \oplus M \mid \alpha(l) = \beta(m)\}$$
(with the obvious maps to L and M).
- In the pullback square
$$\begin{tikzcd} L \prod_N M \arrow[r, "\widehat{\beta}"] \arrow[d, "\widehat{\alpha}"'] & L \arrow[d, "\alpha"] \\ M \arrow[r, "\beta"] & N \end{tikzcd}$$
Show that $\operatorname{Ker} \widehat{\beta} \cong \operatorname{Ker} \beta$.
- In the square in the second point, show that α induces an injective map from $\operatorname{Cok} \widehat{\beta}$ to $\operatorname{Cok} \beta$.

12.09. - 17.09.

Exercise 5. Let A be an abelian group, S and T two subgroups.

- Show that the pullback of

$$\begin{tikzcd} & S \arrow[d, "\text{incl.}"] \\ T \arrow[r, "\text{incl.}"] & A \end{tikzcd}$$

  is given by $S \cap T$.
- Show that the pushout of

$$\begin{tikzcd} A \arrow[r, "\text{proj.}"] \arrow[d, "\text{proj.}"] & A/S \\ A/T & \end{tikzcd}$$

  is given by $A/(S+T)$.

Exercise 6. In an abelian category, consider the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, dashed] & Z \arrow[d, dashed] \arrow[ddl, dotted] \\ W \arrow[r, dashed] \arrow[dr, dotted] & W \coprod_X Y \arrow[r, dashed] & (W \coprod_X Y) \coprod_Y Z \\ & W \coprod_X Z & \end{tikzcd}$$

Show that the "iterated pushout" $(W \coprod_X Y) \coprod_Y Z$ is isomorphic to the pushout along the composition of the horizontal arrows $W \coprod_X Z$.

Exercise 7. In an abelian category, consider a pullback square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

Show that $\operatorname{Ker} f \cong \operatorname{Ker} g$.

19.09. - 24.09.

Exercise 8. In an abelian category, consider a morphism $f \colon X \longrightarrow Y$ with its image $\operatorname{Im} f$.

- Let $e \colon W \twoheadrightarrow X$ be an epimorphism. Show that $\operatorname{Im}(f \circ e) = \operatorname{Im} f$.
- Let $m \colon Y \rightarrowtail Z$ be a monomorphism. Show that $\operatorname{Im}(m \circ f) = \operatorname{Im} f$.
- Assume $f = m \circ e$, with $e \colon X \longrightarrow I$ epi and $m \colon I \longrightarrow Y$ mono. Show that $I = \operatorname{Im} f$.

Exercise 9 (3 × 3 Lemma)**.** Consider the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A \arrow[d] & B \arrow[d] & C \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Show that A, B, and C also form a short exact sequence fitting into the above diagram.

EASIER: In the category $\operatorname{Mod} R$ of modules over a ring.

HARDER: In an arbitrary abelian category.

26.09. - 01.10.

Exercise 10. Consider an abelian category.

- In the situation of the biproduct diagram

$$\begin{tikzcd}
X \arrow[r, bend left, "\iota_X"] & X \oplus Y \arrow[l, bend left, "\pi_X"] \arrow[r, bend right, "\pi_Y"'] & Y \arrow[l, bend right, "\iota_Y"']
\end{tikzcd}$$

  with $\mathrm{id}_X = \pi_X \circ \iota_X$, $\mathrm{id}_Y = \pi_Y \circ \iota_Y$, and $\mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$: Show that π_Y is a cokernel of ι_X.
- Let $f\colon X \longrightarrow Y$ be a split monomorphism. Show that there is an isomorphism $\varphi\colon Y \longrightarrow X \oplus \operatorname{Cok} f$ making the triangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\left(\begin{smallmatrix} \mathrm{id}_X \\ 0 \end{smallmatrix}\right)"'] & Y \arrow[d, "\varphi", "\cong"'] \\
 & X \oplus \operatorname{Cok} f
\end{tikzcd}$$

  commutative.

Exercise 11. In an abelian category, let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be monomorphisms. Show that there is a short exact sequence

$$0 \longrightarrow \operatorname{Cok} f \longrightarrow \operatorname{Cok} g \circ f \longrightarrow \operatorname{Cok} g \longrightarrow 0.$$

REMARK: If we think "$\operatorname{Cok} f = B/A$", "$\operatorname{Cok} g \circ f = C/A$", and "$\operatorname{Cok} g = C/B$", then this exercise gives us the isomorphism theorem $\frac{C/A}{B/A} \cong C/B$.

03.10.-08.10.

Exercise 12. Let X be a poset, and $\mathbb{F}$ a field.

Convince youself that $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$ is an abelian category. (Here all defining constructions take place point-wise.)

For $i \in X$, we consider the special presheaves P_i and I_i given by

$$P_i(j) = \begin{cases} \mathbb{F} & \text{if } j \leqslant i \\ 0 & \text{otherwise} \end{cases} \text{ and } I_i(j) = \begin{cases} k & \text{if } j \geqslant i \\ 0 & \text{otherwise} \end{cases}.$$

Path I:

Verify directly: P_i is projective and I_i is injective in the category $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$.

Path II:

Consider inclusion $\iota\colon \{i\} \hookrightarrow X$. As in Exercise 2 the induced functor

$$\iota^*\colon \operatorname{presh}_{\operatorname{mod}\mathbb{F}} X \longrightarrow \operatorname{presh}_{\operatorname{mod}\mathbb{F}}\{i\} = \operatorname{mod}\mathbb{F}$$

has a left adjoint L and a right adjoint R.

Check that

- ι^* is exact;
- $P_i = L\mathbb{F}$ and $I_i = R\mathbb{F}$;
- $\mathbb{F}$ is both projective and injective in $\operatorname{mod}\mathbb{F}$.

Conclude that P_i is projective and I_i is injective.

Independent of path: For $X = \{a > 0 < b\}$: Find a projective object P and an epimorphism $P \longrightarrow I_0$ in $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$.

Exercise 13. Show that

$$\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m) = \mathbb{Z}/(\gcd(n,m)).$$

Hint: Recall that $(\gcd(m,n)) = (m,n)$.

10.10.-15.10.

Exercise 14. Consider the poset

$$X = \left\{ \begin{array}{ccc} & \omega & \\ a & & b \\ & 0 & \end{array} \right\},$$

and the complex of **Ab**-valued presheaves on it

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dl] \arrow[dr] & & & & \mathbb{Z} \arrow[dl] \arrow[dr] & & & & \mathbb{Z} \arrow[dl] \arrow[dr] & & \\
0 \arrow[r] & 0 \arrow[dr] & & \mathbb{Z} \arrow[dl] \arrow[rr, "\begin{smallmatrix} & 0 & \\ 0 & & 2 \\ & 2 & \end{smallmatrix}"] & & \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] \arrow[rr, "\begin{smallmatrix} & 3 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}"] & & 0 \arrow[dr] & & 0 \arrow[dl] \arrow[r] & 0. \\
 & & \mathbb{Z} & & & & \mathbb{Z} & & & & 0 & &
\end{tikzcd}$$

(Here "2" is short for "the map given by multiplication by 2".)

Calculate all homologies of this complex.

Exercise 15. Let $\mathscr{A}$ be an abelian category. Consider a morphism of complexes over $\mathscr{A}$, of the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Assume this is a quasi-isomorphism.

Show that $B^0 \cong B^{-1} \oplus A^0$, such that the two non-zero maps above are the canonical inclusions.

17.10.-22.10.

Exercise 16. Consider the X as below, and a field $\mathbb{F}$. Calculate a projective resolution of I_ω in $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$ (see Exercise 12).

(1) For $X = \{0 < \omega\}$;

(2) For $X = \left\{ \begin{tikzcd}[column sep=small, row sep=small] & \omega & \\ a \arrow[ur, dash] & & b \arrow[ul, dash] \\ & 0 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \right\}$.

Exercise 17. Let $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes. Show that $B^\bullet \longrightarrow \operatorname{Cone}(f^\bullet)$ is a *weak cokernel* of $f^\bullet$ in the homotopy category $\mathbf{K}(\mathscr{A})$.

That is, the composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow \operatorname{Cone}(f^\bullet)$ is zero, and for any $g^\bullet \colon B^\bullet \longrightarrow C^\bullet$ such that $g^\bullet \circ f^\bullet = 0$ in $\mathbf{K}(\mathscr{A})$ there is a (not necessarily unique) factorization

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \arrow[r] \arrow[dr, "g^\bullet"'] & \operatorname{Cone}(f^\bullet) \arrow[d, dashed] \\
& & C^\bullet
\end{tikzcd}$$

24.10.-29.10.

Exercise 18. Let $\mathbb{F}$ be a field. Calculate

$$\operatorname{Ext}^n_{\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X}(-, P_0)(I_\omega)$$

for all n and both choices of X as in Exercise 16

Exercise 19 (Balancing Ext)**.** Let $\mathscr{A}$ be an abelian category with enough projectives.

(1) For a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$, and an object X, show that there is a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^n_{\mathscr{A}}(-, A)(X) \longrightarrow \operatorname{Ext}^n_{\mathscr{A}}(-, B)(X) \longrightarrow \operatorname{Ext}^n_{\mathscr{A}}(-, C)(X)$$
$$\longrightarrow \operatorname{Ext}^{n+1}_{\mathscr{A}}(-, A)(X) \longrightarrow \operatorname{Ext}^{n+1}_{\mathscr{A}}(-, B)(X) \longrightarrow \operatorname{Ext}^{n+1}_{\mathscr{A}}(-, C)(X) \longrightarrow \cdots$$

(2) Show that there is dimension shift with respect to the "inner" argument, that is provided we have a short exact sequence

$$0 \longrightarrow A \longrightarrow I \longrightarrow \mho A$$

with I injective, then

$$\operatorname{Ext}^n_{\mathscr{A}}(-,A)(X) \cong \begin{cases} \operatorname{Ext}^{n-1}_{\mathscr{A}}(-,\mho A)(X) & \text{if } n \geqslant 2 \\ \operatorname{Cok}[\operatorname{Hom}_{\mathscr{A}}(X,I) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(X,\mho A)] & \text{if } n = 1 \end{cases}$$

(3) Assume now that $\mathscr{A}$ in addition has enough injectives. Show that

$$\operatorname{Ext}^n_{\mathscr{A}}(-,A)(X) \cong \operatorname{Ext}^n_{\mathscr{A}}(X,-)(A).$$

31.10.-05.11.

Exercise 20 (A small spectral sequence)**.** Consider a double complex $X^{\bullet,\bullet}$ with $X^{m,n} = 0$ unless $m,n \in \{-1,0\}$ – that is essentially a commutative square

$$\begin{tikzcd} X^{-1,-1} \arrow[r, "d_{\mathrm{h}}^{-1,-1}"] \arrow[d, "d_{\mathrm{v}}^{-1,-1}"'] & X^{0,-1} \arrow[d, "d_{\mathrm{v}}^{0,-1}"] \\ X^{-1,0} \arrow[r, "d_{\mathrm{h}}^{-1,0}"] & X^{0,0} \end{tikzcd}$$

We consider the kernels and cokernels of the horizontal maps, and denote by $k\colon \operatorname{Ker} d_{\mathrm{h}}^{-1,-1} \longrightarrow \operatorname{Ker} d_{\mathrm{h}}^{-1,0}$ and $c\colon \operatorname{Cok} d_{\mathrm{h}}^{-1,-1} \longrightarrow \operatorname{Cok} d_{\mathrm{h}}^{-1,0}$ the kernel and cokernel morphism, respectively.

Show that

- $\mathrm{H}^{-2}(\operatorname{Tot}(X^{\bullet,\bullet})) = \operatorname{Ker} k$;
- There is a short exact sequence $\operatorname{Cok} k \rightarrowtail \mathrm{H}^{-1}(\operatorname{Tot}(X^{\bullet,\bullet})) \twoheadrightarrow \operatorname{Ker} c$;
- $\mathrm{H}^{0}(\operatorname{Tot}(X^{\bullet,\bullet})) = \operatorname{Cok} c$.

Exercise 21. Let $\mathscr{A}$ be an abelian category with enough projectives. Consider two short exact sequences $C \rightarrowtail F \twoheadrightarrow B$ and $B \rightarrowtail E \twoheadrightarrow A$.

Assume that $\operatorname{Ext}^2_{\mathscr{A}}(A,C) = 0$.

Show that there is an object X completing the following diagram as indicated by the dashed arrows.

$$\begin{tikzcd} C \arrow[r, tail] \arrow[d, equal] & F \arrow[r, two heads] \arrow[d, tail, dashed] & B \arrow[d, tail] \\ C \arrow[r, tail, dashed] & X \arrow[r, two heads, dashed] \arrow[d, two heads, dashed] & E \arrow[d, two heads] \\ & A \arrow[r, equal] & A \end{tikzcd}$$

(That is, in the resulting diagram all squares commute and all rows and columns are short exact sequences.)

07.11.-12.11.

Exercise 22. Let $\mathscr{A}$ be an abelian category. Let $A^\bullet \in \mathbf{C}(\mathscr{A})$. Show that $A^\bullet \cong 0$ in $\mathbf{K}(\mathscr{A})$ if and only if $A^\bullet$ is isomorphic to a complex of the form

$$\cdots \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} B^{-1} \oplus B^0 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} B^0 \oplus B^1 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} B^1 \oplus B^2 \xrightarrow{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)} \cdots.$$

Hint: For the forward direction, consider the short exact sequences

$$0 \longrightarrow \mathrm{B}^n(A^\bullet) \longrightarrow A^n \longrightarrow \mathrm{B}^{n+1}(A^\bullet) \longrightarrow 0.$$

Exercise 23. Let $\mathscr{T}$ be a triangulated category. Assume there is a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} X[1],$$

that is a triangle where the last map is zero.

Show that $X \longrightarrow Y \longrightarrow Z$ is a split short exact sequence. (That is there are maps $\tilde{f} \colon Y \longrightarrow X$ and $\tilde{g} \colon Z \longrightarrow Y$ such that $\mathrm{id}_X = \tilde{f} \circ f$, $\mathrm{id}_Y = f \circ \tilde{f} + \tilde{g} \circ g$, and $\mathrm{id}_Z = g \circ \tilde{g}$.)

14.11.-19.11.

Exercise 24. Let $\mathscr{A}$ be a semisimple abelian category. Show that any object in $\mathbf{K}(\mathscr{A})$ is isomorphic to its homology. That is, given a complex $A^\bullet$ show

$$A^\bullet \cong [\cdots \xrightarrow{0} \mathrm{H}^{-1}(A^\bullet) \xrightarrow{0} \mathrm{H}^0(A^\bullet) \xrightarrow{0} \mathrm{H}^1(A^\bullet) \xrightarrow{0} \cdots].$$

Exercise 25. Let $\mathscr{A}$ be abelian. Let $A^\bullet$ be a complex concentrated in negative degrees ($A^n = 0 \ \forall n \geqslant 0$), and $B^\bullet$ be a complex concentrated in non-negative degrees ($B^n = 0 \ \forall n < 0$).

Show that $\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A^\bullet, B^\bullet) = 0$.