EXERCISES

21.08.

Exercise 1.
- Describe which morphisms in **Set** are monomorphisms, epimorphisms, split monomorphisms, and split epimorphisms.
- Describe which morphisms in **Top** are monomorphisms and which morphism are epimorphisms. Find an example of a morphism that is both a monomorphism and an epimorphism, but not an isomorphism.
- Show that in the category **Ring**, the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism.

Exercise 2. Which of the following functors are full? faithful? dense?
- the natural inclusion $\mathbf{Ab} \longrightarrow \mathbf{Gp}$;
- forgetting the topology $\mathbf{Top} \longrightarrow \mathbf{Set}$;
- the Hom-functor $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/(2), -)\colon \mathbf{Ab} \longrightarrow \mathbf{Set}$.

Exercise 3. Let X be the poset given by the Hasse diagram $\begin{array}{ccc} a & & b \\ & \diagdown\ \diagup & \\ & 0 & \end{array}$ that is $a \geqslant 0$ and $b \geqslant 0$ with a and b incomparable.
- Determine all objects $F \in \mathrm{presh}_{\mathbf{Set}}\, X$ such that $F(i) \in \{\emptyset, \{\star\}\}\ \forall i \in X$.
- Which of the presheaves determined above are isomorphic to a presheaf of the form $\mathrm{Hom}_{\mathscr{C}_X}(-, i)$ for some $i \in X$?

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Exercise 4. Recall that for a group G, we denote by G^{op} the opposite group, that is the group with the same elements as G, but multiplication given by $g \cdot_{\mathrm{op}} h = h \cdot g$.
- Show that this construction defines a (covariant!) functor $\mathbf{Gp} \longrightarrow \mathbf{Gp}$.
- Any group G is isomorphic to its opposite group: An isomorphism is given by $g \longmapsto g^{-1}$. Investigate if this defines a natural isomorphism $\mathrm{id}_{\mathbf{Gp}} \longrightarrow -^{\mathrm{op}}$.

Exercise 5. Let G be a non-trivial group. We can consider the category $\mathscr{C}_G$ having only one object $\star$, with $\mathrm{Hom}_{\mathscr{C}_G}(\star, \star) = G$, and composition of morphisms given by group multiplication. (Convince yourself that this is a category.)

Consider the following two functors $\mathtt{F} = \mathrm{Hom}_{\mathscr{C}_G}(\star, -)$ and $\mathtt{H}\colon \mathscr{C}_G \longrightarrow \mathbf{Set}$ given by $\mathtt{H}(\star) = G$ and $\mathtt{H}(g) = 1_G$.

Show that the functors $\mathtt{F}$ and $\mathtt{H}$ agree on all (i.e. the one) objects, but are not naturally isomorphic.

Exercise 6. Let $X = \{a, b, c\}$ with the preorder given $a \leqslant a$, $a \leqslant b$, $a \leqslant c$, $b \leqslant a$, $b \leqslant b$, $b \leqslant c$, $c \leqslant c$. (So a and b violate anti-symmetry). Let $Y = \{1, 2\}$ with the natural poset structure (i.e. $1 \leqslant 2$). Show that the poset categories $\mathscr{C}_{(X,\leqslant)}$ and $\mathscr{C}_{(Y,\leqslant)}$ are equivalent.

More generally, given an arbitrary preordered set X, find a poset Y such that the cateories $\mathscr{C}_{(X,\leqslant)}$ and $\mathscr{C}_{(Y,\leqslant)}$ are equivalent.

0.1. **04.09.**

Exercise 7. Find left adjoints to the functors
- $\mathtt{forget}\colon \mathbf{Ring} \longrightarrow \mathbf{Rng}$, the forgetful functor from rings with multiplicative unit to rings without multiplicative unit.

- `forget`: $\mathbf{Ring}_* \longrightarrow \mathbf{Ring}$, where $\mathbf{Ring}_*$ is the category of "pointed rings", that is pairs (R, r) of a Ring R and an element r, and morphisms being ring homomorphisms which send the distinguished element of the first ring to the distinguished element of the second ring.

Find the unit and counit maps for both the above adjunctions.

Exercise 8. In the category $\mathbf{Ab}$

- Show that the pullback of

$$\begin{tikzcd} & L \arrow[d, "\alpha"] \\ M \arrow[r, "\beta"] & N \end{tikzcd}$$

is given by

$$L \prod_N M = \{(l, m) \in L \oplus M \mid \alpha(l) = \beta(m)\}$$

(with the obvious maps to L and M).

- Show that the pushout of

$$\begin{tikzcd} L \arrow[r, "\beta"] \arrow[d, "\alpha"'] & M \\ N & \end{tikzcd}$$

is given by

$$M \coprod_L N = M \oplus N / \{(\beta(\ell), -\alpha(\ell)) \mid \ell \in L\}.$$

Exercise 9. Let X be any poset, and F a $\mathbf{Set}$-valued presheaf on X. Show

- that the limit $\varprojlim F$ exists;
- that the colimit $\varinjlim F$ exists.

Hint: Construct them explicitly, starting with product and coproduct, respectively.

10.09.

Exercise 10. Let A be an abelian group, S and T two subgroups.

- Show that the pullback of

$$\begin{tikzcd} & S \arrow[d, "\text{incl.}"] \\ T \arrow[r, "\text{incl.}"] & A \end{tikzcd}$$

is given by $S \cap T$.

- Show that the pushout of

$$\begin{tikzcd} A \arrow[r, "\text{proj.}"] \arrow[d, "\text{proj.}"] & A/S \\ A/T & \end{tikzcd}$$

is given by $A/(S+T)$.

Exercise 11. In the category **Ab**, consider a pullback square

$$\begin{tikzcd} L \prod_N M \arrow[r, "\hat{\beta}"] \arrow[d, "\hat{\alpha}"] & L \arrow[d, "\alpha"] \\ M \arrow[r, "\beta"] & N \end{tikzcd}$$

Show that $\operatorname{Ker} \hat{\beta} \cong \operatorname{Ker} \beta$ and $\operatorname{Ker} \hat{\alpha} \cong \operatorname{Ker} \alpha$.

HINT: Recall the explicit description of the pullback in Exercise 8.

Exercise 12. Let $\mathscr{A}$ be a pre-abelian category, and X be a finite poset. Show that any $F \in \operatorname{presh}_{\mathscr{A}} X$ has a limit and a colimit.

Exercise 13. Let $\mathscr{A}$ be a pre-abelian category. Show that the following are equivalent: ★

- $\mathscr{A}$ is abelian;
- every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.

HINT: First show that if a monomorphism is a kernel of some morphism, then it is in fact a kernel of its cokernel.

18.09.

Exercise 14. Consider an additive category.

- In the situation of the biproduct diagram

$$\begin{tikzcd} X \arrow[r, bend left, "\iota_X"] & X \oplus Y \arrow[l, bend left, "\pi_X"] \arrow[r, bend right, "\pi_Y"'] & Y \arrow[l, bend right, "\iota_Y"'] \end{tikzcd}$$

  with $\mathrm{id}_X = \pi_X \circ \iota_X$, $\mathrm{id}_Y = \pi_Y \circ \iota_Y$, and $\mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y$: Show that π_Y is a cokernel of ι_X.
- Let $f\colon X \longrightarrow Y$ be a split monomorphism, and assume f has a cokernel. Show that there is an isomorphism $\varphi\colon Y \longrightarrow X \oplus \operatorname{Cok} f$ making the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\left(\begin{smallmatrix} \mathrm{id}_X \\ 0 \end{smallmatrix}\right)"'] & Y \arrow[d, "\varphi", "\cong"'] \\ & X \oplus \operatorname{Cok} f \end{tikzcd}$$

  commutative.

Exercise 15.
- Consider the poset

$$X = \left\{ \begin{tikzcd}[column sep=small, row sep=small] & \omega \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd} \right\},$$

and the morphism of **Ab**-valued presheaves on it

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dl] \arrow[dr] & & & \mathbb{Z} \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & & \mathbb{Z} \arrow[dl] \arrow[r, "{\begin{smallmatrix} & 0 & \\ 0 & & 2 \\ & 0 & \end{smallmatrix}}"] & \mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] . \\
 & \mathbb{Z} & & & 0 &
\end{tikzcd}$$

(Here "2" is short for "the map given by multiplication by 2".) Check that this is a morphism. Calculate the kernel, image, and cokernel of this morphism.

- Convince yourself that in general, for an abelian category $\mathscr{A}$ and a poset X, the presheaf category $\operatorname{presh}_{\mathscr{A}} X$ is abelian again.

Exercise 16 (3 × 3 Lemma)**.** Consider the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & A \arrow[d] & B \arrow[d] & C \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Show that A, B, and C also form a short exact sequence fitting into the above diagram.

★ **Exercise 17** (Salamander lemma)**.** Consider the following diagram in $\mathbf{Mod}R$, where $\gamma\beta = 0$ and $\delta\gamma = 0$.

$$\begin{tikzcd}
 & A \arrow[d, "\alpha"] & & \\
B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & D \arrow[r, "\delta"] \arrow[d, "\epsilon"] & E \\
 & & F &
\end{tikzcd}$$

Show that the sequence

$$\operatorname{Ker}\gamma\alpha \longrightarrow \frac{\operatorname{Ker}\gamma}{\operatorname{Im}\beta} \longrightarrow \frac{\operatorname{Ker}\epsilon\gamma}{\operatorname{Im}\alpha + \operatorname{Im}\beta} \longrightarrow \frac{\operatorname{Ker}\delta \cap \operatorname{Ker}\epsilon}{\operatorname{Im}\gamma\alpha} \longrightarrow \frac{\operatorname{Ker}\delta}{\operatorname{Im}\gamma} \longrightarrow \frac{F}{\operatorname{Im}\epsilon\gamma}$$

is exact.

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Exercise 18. Let X be a poset, and $\mathbb{F}$ a field. By Exercise 15 the category $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$ is abelian.

For $i \in X$, we consider the special presheaves P_i and I_i given by

$$P_i(j) = \begin{cases} \mathbb{F} & \text{if } j \leqslant i \\ 0 & \text{otherwise} \end{cases} \text{ and } I_i(j) = \begin{cases} k & \text{if } j \geqslant i \\ 0 & \text{otherwise} \end{cases}.$$

Path I:
Verify directly: P_i is projective and I_i is injective in the category $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$.

Path II:
Consider inclusion $\iota\colon \{i\} \hookrightarrow X$. Show that the induced functor

$$\iota^*\colon \operatorname{presh}_{\operatorname{mod}\mathbb{F}} X \longrightarrow \operatorname{presh}_{\operatorname{mod}\mathbb{F}}\{i\} = \operatorname{mod}\mathbb{F}$$

has a left adjoint L and a right adjoint R.

Check that

- ι^* is exact;
- $P_i = L\mathbb{F}$ and $I_i = R\mathbb{F}$;
- $\mathbb{F}$ is both projective and injective in $\operatorname{mod}\mathbb{F}$.

Conclude that P_i is projective and I_i is injective.

Independent of path: For $X = \{a > 0 < b\}$: Find a projective object P and an epimorphism $P \longrightarrow I_0$ in $\operatorname{presh}_{\operatorname{mod}\mathbb{F}} X$.

Exercise 19. Show that

$$\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m) = \mathbb{Z}/(\gcd(n,m)).$$

Hint: Recall that $(\gcd(m,n)) = (m,n)$.

Exercise 20. Let $\mathscr{A}$ be an abelian category, and A an object in $\mathscr{A}$. Convince ★
yourself that $\operatorname{Hom}_{\mathscr{A}}(A,-)$ defines a functor $\mathscr{A} \longrightarrow \operatorname{Mod} R$, where $R = \operatorname{End}_{\mathscr{A}}(A)$.

Now assume that for any object $X \in \mathscr{A}$, the R-module $\operatorname{Hom}_{\mathscr{A}}(A,X)$ is finitely generated. Show (without using Freyd-Mitchell), that $\operatorname{Hom}_{\mathscr{A}}(A,-)\colon \mathscr{A} \longrightarrow \operatorname{mod} R$ has a left adjoint.

Hints:

- Show that $\operatorname{Hom}_{\mathscr{A}}(A,-)$ induces an equivalence between the subcategories $\{A^n \mid n \in \mathbb{N}\} \subseteq \mathscr{A}$ and $\{R^n \mid n \in \mathbb{N}\} \subseteq \operatorname{mod} R$.
- Show that $\operatorname{Hom}_{\mathscr{A}}(A,X)$ is finitely presented as R-module.

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Exercise 21. Let $A^\bullet \in \mathbf{C}(\mathscr{A})$ be a complex such that $\mathtt{H}^i(A^\bullet) = 0$ for all negative i. Show that there is a complex $B^\bullet$ such that $B^i = 0$ for all negative i, and a quasi-isomorphism $A^\bullet \longrightarrow B^\bullet$.

Hint: You can take $B^i = A^i$ for all positive i. What is a good choice for B^0?

Exercise 22. Let $f^\bullet\colon A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes. Show that $B^\bullet \longrightarrow \operatorname{Cone}(f^\bullet)$ is a *weak cokernel* of $f^\bullet$ in the homotopy category $\mathbf{K}(\mathscr{A})$.

That is, the composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow \mathrm{Cone}(f^\bullet)$ is zero, and for any $g^\bullet \colon B^\bullet \longrightarrow C^\bullet$ such that $g^\bullet \circ f^\bullet = 0$ in $\mathbf{K}(\mathscr{A})$ there is a (not necessarily unique) factorization

$$\begin{tikzcd} A^\bullet \arrow[r, "f^\bullet"] & B^\bullet \arrow[r] \arrow[dr, "g^\bullet"'] & \mathrm{Cone}(f^\bullet) \arrow[d, dashed] \\ & & C^\bullet \end{tikzcd}$$

Exercise 23. Calculate
- $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/(a), \mathbb{Z}/(b))$ for alle $a, b, n \in \mathbb{N}$;
- $\operatorname{Ext}^n_{\mathrm{presh}_{\mathrm{mod}\,\mathbb{F}}\{1<2\}}(I_2, P_1)$ for all $n \in \mathbb{N}$;
- $\operatorname{Ext}^n_{\mathrm{presh}_{\mathrm{mod}\,\mathbb{F}} X}(I_\omega, P_0)$ for all $n \in \mathbb{N}$, where X is the poset from Exercise 15.

★ **Exercise 24.** Let $\mathscr{A}$ be an abelian category with enough injectives. We consider the category of morphisms in $\mathscr{A}$,

$$\mathbf{mor}(\mathscr{A}) = \mathrm{presh}_{\mathscr{A}}\{1 < 2\}.$$

- Convince yourself that Ker defines a left exact functor $\mathbf{mor}(\mathscr{A}) \longrightarrow \mathscr{A}$.
- Find out what the right derived functors $\mathbb{R}^n \operatorname{Ker}$ are.

Hint: First consider the case that the morphism in question is an epimorphism. Then generalize to arbitrary morphisms using a short exact sequence in $\mathbf{mor}(\mathscr{A})$ where the other two objects are epimorphisms.

23.10.

Exercise 25. Let R as below, and S be the R-module which is $\mathtt{F}$ as $\mathbb{F}$-vector space, with all variables acting as 0. Calculate all $\operatorname{Ext}^n_R(S, S)$ for $n \geqslant 0$.
- $R = \mathbb{F}[X]$;
- $R = \mathbb{F}[X]/(X^3)$;
- $R = \mathbb{F}[X, Y]$;
- $R = \mathbb{F}[X, Y]/(XY)$.

Exercise 26. Find short exact sequences representing all elements of $\operatorname{YExt}^1_{\mathbf{Ab}}(\mathbb{Z}/(4), \mathbb{Z}/4)$.

Exercise 27. Let $\mathscr{A}$ be an abelian category. Let $A^\bullet \in \mathbf{C}(\mathscr{A})$. Show that $A^\bullet \cong 0$ in $\mathbf{K}(\mathscr{A})$ if and only if $A^\bullet$ is isomorphic to a complex of the form

$$\cdots \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} B^{-1} \oplus B^0 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} B^0 \oplus B^1 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} B^1 \oplus B^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} \cdots.$$

Hint: For the “only if” direction, consider the short exact sequences

$$0 \longrightarrow \mathrm{B}^n(A^\bullet) \longrightarrow A^n \longrightarrow \mathrm{B}^{n+1}(A^\bullet) \longrightarrow 0.$$

★ **Exercise 28** (A short spectral sequence)**.** Consider a double complex $X^{\bullet,\bullet}$ with $X^{m,n} = 0$ unless $m, n \in \{-1, 0\}$ – that is essentially a commutative square

$$\begin{tikzcd} X^{-1,-1} \arrow[r, "d_{\mathrm{h}}^{-1,-1}"] \arrow[d, "d_{\mathrm{v}}^{-1,-1}"'] & X^{0,-1} \arrow[d, "d_{\mathrm{v}}^{0,-1}"] \\ X^{-1,0} \arrow[r, "d_{\mathrm{h}}^{-1,0}"] & X^{0,0} \end{tikzcd}$$

We consider the kernels and cokernels of the horizontal maps, and denote by $k\colon \operatorname{Ker} d_{\mathrm{h}}^{-1,-1} \longrightarrow \operatorname{Ker} d_{\mathrm{h}}^{-1,0}$ and $c\colon \operatorname{Cok} d_{\mathrm{h}}^{-1,-1} \longrightarrow \operatorname{Cok} d_{\mathrm{h}}^{-1,0}$ the kernel and cokernel morphism, respectively.

Show that

- $\mathrm{H}^{-2}(\mathrm{Tot}(X^{\bullet,\bullet})) = \operatorname{Ker} k$;
- There is a short exact sequence $\operatorname{Cok} k \rightarrowtail \mathrm{H}^{-1}(\mathrm{Tot}(X^{\bullet,\bullet})) \twoheadrightarrow \operatorname{Ker} c$;
- $\mathrm{H}^{0}(\mathrm{Tot}(X^{\bullet,\bullet})) = \operatorname{Cok} c$.

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Exercise 29. Let $R = \mathbb{F}[X,Y]/(XY)$ for some field $\mathbb{F}$, and $M = R/(X)$. Calculate $\operatorname{Ext}^n_R(M,M)$ for all $n \in \mathbb{N}$.

Exercise 30. Let $R = \mathbb{F}[X,Y]/(XY)$ for some field $\mathbb{F}$. Consider the double complex $X^{\bullet,\bullet}$ given by

$$X^{m,n} = R, \quad d_{\mathrm{h}}^{m,n} = d_{\mathrm{v}}^{m,n} = \begin{cases} X & \text{if } m+n \text{ even} \\ Y & \text{if } m+n \text{ odd} \end{cases}.$$

Show that all rows and all columns of $X^{\bullet,\bullet}$ are exact, but its total complex is not exact.

Exercise 31. Let R be a ring, and $X^{\bullet,\bullet}$ a double complex of R-modules. assume that $X^{m,n} = 0$ whenever $n > 0$. (That is $X^{\bullet,\bullet}$ is concentrated on the upper half plane.) Show that if all rows of $X^{\bullet,\bullet}$ are exact then so is its total complex.

Exercise 32. Let $\mathscr{A}$ be an abelian category with enough projectives. Consider two short exact sequences $C \rightarrowtail F \twoheadrightarrow B$ and $B \rightarrowtail E \twoheadrightarrow A$.

Assume that $\operatorname{Ext}^2_{\mathscr{A}}(A,C) = 0$.

Show that there is an object X completing the following diagram as indicated by the dashed arrows.

$$\begin{tikzcd}
C \arrow[r, tail] \arrow[d, equal] & F \arrow[r, two heads] \arrow[d, tail, dashed] & B \arrow[d] \\
C \arrow[r, tail, dashed] & X \arrow[r, two heads, dashed] \arrow[d, two heads, dashed] & E \arrow[d, two heads] \\
& A \arrow[r, equal] & A
\end{tikzcd}$$

(That is, in the resulting diagram all squares commute and all rows and columns are short exact sequences.)

06.11.

Exercise 33. Let $\mathscr{A}$ be an abelian category with enough projectives. Show that $\operatorname{gl.dim} \mathscr{A} \leqslant 2$ if and only if any morphism between projectives has a projective kernel.

Exercise 34. Let $\mathscr{A}$ ba an abelian category with enough projectives. Show that $\star$

$$\operatorname{gl.dim} \operatorname{presh}_{\mathscr{A}}\{1 < 2\} = \operatorname{gl.dim} \mathscr{A} + 1.$$

Exercise 35. Let $\mathscr{T}$ be a triangulated category, and $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ a distinguished triangle. Assume the map $Z \longrightarrow X[1]$ is 0. Show that the (rest of the) triangle then is a split short exact sequence.

Exercise 36. Let $\mathscr{T}$ be a triangulated category. Assume that $\mathscr{T}$ is in addition abelian. Show that $\mathscr{T}$ is semisimple.

13.11.

Exercise 37. Let $\mathscr{A}$ be an abelian category. Show that any complex is isomorphic to its homology in $\mathbf{K}(\mathscr{A})$ if and only if $\mathscr{A}$ is semisimple.

(Here the homology of a complex $X^\bullet$ is considered as the complex $\cdots \xrightarrow{0} \mathrm{H}^{-1}(X^\bullet) \xrightarrow{0} \mathrm{H}^0(X^\bullet) \xrightarrow{0} \mathrm{H}^1(X^\bullet) \longrightarrow \cdots$.)

Exercise 38. Let $\mathscr{T}$ be a triangulated category, and $C \longrightarrow F \longrightarrow B \xrightarrow{f} C[1]$ and $B \longrightarrow E \longrightarrow A \xrightarrow{g} B[1]$ two distinguished triangles. Assume the composition $f[1] \circ g \colon A \longrightarrow C[2]$ vanishes. Show that there is an object X and morphisms as indicated by the dashed arrows below, such that the diagram commutes and the new row and new column are distinguished triangles too.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
C \arrow[r, dashed] & X \arrow[r, dashed] \arrow[d, dashed] & E \arrow[r, dashed] \arrow[d] & C[1] \\
& A \arrow[r, equal] \arrow[d, dashed] & A \arrow[d] & \\
& F[1] \arrow[r] & B[1] &
\end{tikzcd}$$

(Compare to Exercise 32.)

★ **Exercise 39.** Let $\mathscr{T}$ be a triangulated category, and $\mathscr{U}$ be a triangulated subcategory. (That is a full subcategory closed under [1] and [−1], and such that the cone of any morphism in the subcategory is in $\mathscr{U}$ again.)

Let $\mathscr{S}$ be the collection of all morphisms in $\mathscr{T}$, whose cone lies in $\mathscr{U}$.

Show that (up to set theoretical issues) one can define a triangulated category $\mathscr{S}^{-1}\mathscr{T}$ making all morphisms in $\mathscr{S}$ invertible in the same way as we defined the derived category in the lectures.