Homological Algebra

Steffen Oppermann

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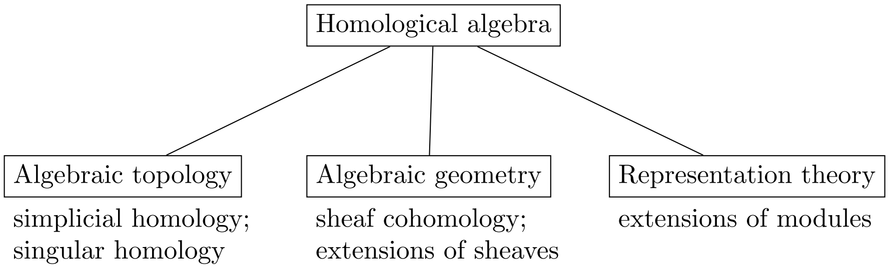
Connections



Example 1.1. Let $f\colon A \longrightarrow B$ be a surjective map, and $g\colon X \longrightarrow B$ any map. One may ask if there is a map $h\colon X \longrightarrow A$ such that $g = f \circ h$.

- If we are just talking about sets, and maps, the answer is “yes”: for any $x \in X$ pick a preimage of $g(x)$.
- If we are talking about vector spaces and linear maps the answer is also “yes”: find a basis of X, then pick a preimage of $g(x)$ for any basis element x.
- If we are talking about abelian groups and linear maps the answer is “sometimes”:
  - Let $A = \mathbb{Z}/(4)$, $B = X = \mathbb{Z}/(2)$, and let f be the natural projection and g the identity. Then there is no linear map h such that $g = f \circ h$.
  - Let $A = \mathbb{Z}/(6)$, $B = X = \mathbb{Z}/(2)$, and let f be the natural projection and g the identity. Then there is a linear map h such that $g = f \circ h$, given by sending the residue class of 1 to the residue class of 3.

We will see: The obstruction to finding h is measure by the group Ext^1. (In the examples above we have $\mathrm{Ext}^1_{\text{vector spaces}} = 0$, and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/(2)\mathbb{Z}/3) = 0$, but $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \neq 0$.)

Chapter I

General categories

2 Definition

In many situations in algebra (but also other parts of mathematics) we consider some type of structures, say vector spaces, groups, rings, or similar. Typically these are sets with some additional properties or structure. When studying these kind of situations, there are two basic ingredients: We study the objects having the desired structure themselves, and we study maps between objects which preserve the structure (i.e. linear maps, group homomorphisms, ring homomorphisms, ...). The concept of a *category* axiomatizes this.

Definition 2.1. A *category* $\mathscr{C}$ consists of

- a class of *objects* $\mathfrak{Ob}\,\mathscr{C}$;
- for any two objects X and Y a set of *morphisms* $\operatorname{Hom}_{\mathscr{C}}(X, Y)$;
- for any three objects X, Y, and Z, a multiplication map

$$\operatorname{Hom}_{\mathscr{C}}(Y, Z) \times \operatorname{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X, Z) \colon (f, g) \longmapsto f \circ g.$$

such that

- for any object X there is a morphism $\mathrm{id}_X \in \operatorname{Hom}_{\mathscr{C}}(X, X)$ such that

$$\forall Y \in \mathfrak{Ob}\,\mathscr{C}\ \forall f \in \operatorname{Hom}_{\mathscr{C}}(X, Y) \colon f \circ \mathrm{id}_X = f,$$
$$\forall Y \in \mathfrak{Ob}\,\mathscr{C}\ \forall f \in \operatorname{Hom}_{\mathscr{C}}(Y, X) \colon \mathrm{id}_X \circ f = f.$$

- multiplication is associative, that is for any objects X, Y, Z, and W and $f \in \operatorname{Hom}_{\mathscr{C}}(X,Y), g \in \operatorname{Hom}_{\mathscr{C}}(Y,Z)$, and $h \in \operatorname{Hom}_{\mathscr{C}}(Z,W)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Remark 2.2. Often $\operatorname{Hom}_{\mathscr{C}}$ is just all maps with some additional nice property.

Example 2.3.

- $\mathscr{C} = \mathbf{Set}$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\,\mathbf{Set} = \{\text{sets}\}, \text{ and} \\ &\operatorname{Hom}_{\mathbf{Set}}(X,Y) = \{\text{maps form } X \text{ to } Y\}. \end{aligned}$$

- $\mathscr{C} = \mathbf{Gp}$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\,\mathbf{Gp} = \{\text{groups}\}, \text{ and} \\ &\operatorname{Hom}_{\mathbf{Gp}}(G,H) = \{\text{group homomorphisms } G \text{ to } H\}. \end{aligned}$$

- $\mathscr{C} = \mathbf{Ab}$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\,\mathbf{Ab} = \{\text{Abelian groups}\}, \text{ and} \\ &\operatorname{Hom}_{\mathbf{Ab}}(G,H) = \{\text{group homomorphisms } G \text{ to } H\}. \end{aligned}$$

- $\mathscr{C} = \mathbf{Top}$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\,\mathbf{Top} = \{\text{topological spaces}\}, \text{ and} \\ &\operatorname{Hom}_{\mathbf{Top}}(X,Y) = \{\text{continuous maps } X \text{ to } Y\}. \end{aligned}$$

- For a ring R, $\mathscr{C} = \operatorname{Mod} R$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\operatorname{Mod} R = \{\text{right } R\text{-modules}\}, \text{ and} \\ &\operatorname{Hom}_{\operatorname{Mod} R}(M,N) = \{R\text{-module homomorphisms } M \text{ to } N\}. \end{aligned}$$

- For a ring R, $\mathscr{C} = \operatorname{mod} R$:

$$\begin{aligned} &\mathcal{O}\mathfrak{b}\operatorname{mod} R = \{\text{finitely generated right } R\text{-modules}\}, \text{ and} \\ &\operatorname{Hom}_{\operatorname{mod} R}(M,N) = \operatorname{Hom}_{\operatorname{Mod} R}(M,N). \end{aligned}$$

Observation 2.4. For a category $\mathscr{C}$, one can define the *opposite category* $\mathscr{C}^{\mathrm{op}}$ by $\mathcal{O}\mathfrak{b}\,\mathscr{C}^{\mathrm{op}} = \mathcal{O}\mathfrak{b}\,\mathscr{C}$, and $\operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(X,Y) = \operatorname{Hom}_{\mathscr{C}}(Y,X)$, together with the multiplication rule $f \circ_{\mathscr{C}^{\mathrm{op}}} g = g \circ_{\mathscr{C}} f$.

One simple toy example of categories is the following

Construction 2.5. Let $(X, \leqslant)$ be a poset. The *poset category* $\mathscr{C}_{(X,\leqslant)}$ is defined by

$$\begin{aligned} \mathfrak{Ob}\,\mathscr{C}_{(X,\leqslant)} &= X, \text{ and} \\ \operatorname{Hom}_{\mathscr{C}_{(X,\leqslant)}}(x,y) &= \begin{cases} \{\iota_x^y\} & \text{if } x \leqslant y \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned}$$

where $\iota_y^z \circ \iota_x^y = \iota_x^z$ whenever $x \leqslant y \leqslant z$.

More generally, this construction works for a preordered set, that is a set with an order that is not necessarily anti-symmetric.

Definition 2.6. A *subcategory* $\mathscr{S}$ of a category $\mathscr{C}$ consist of

- A subclass $\mathfrak{Ob}\,\mathscr{S}$ of $\mathfrak{Ob}\,\mathscr{C}$;
- for every $S, T \in \mathfrak{Ob}\,\mathscr{S}$, a subset $\operatorname{Hom}_{\mathscr{S}}(S,T) \subseteq \operatorname{Hom}_{\mathscr{C}}(S,T)$;

such that the identity on any object in $\mathscr{S}$ is a morphism in $\mathscr{S}$, and compositions of morphisms in $\mathscr{S}$ are morphisms in $\mathscr{S}$ again.

The subcategory $\mathscr{S} \subseteq \mathscr{C}$ is called *full* if for all $S, T \in \mathfrak{Ob}\,\mathscr{S}$, $\operatorname{Hom}_{\mathscr{S}}(S,T) = \operatorname{Hom}_{\mathscr{C}}(S,T)$.

Example 2.7.

- **Ab** is a full subcategory of **Gp**.
- For a poset $(X, \leqslant)$, and $Y \subseteq X$ with induced poset structure, the poset category $\mathscr{C}_{(Y,\leqslant)}$ is a full subcategory of $\mathscr{C}_{(X,\leqslant)}$.

Definition 2.8. A morphism $f \in \operatorname{Hom}_{\mathscr{C}}(X,Y)$ is called

- *monomorphism*, if for any W and any $g, h \in \operatorname{Hom}_{\mathscr{C}}(W,X)$ such that $f \circ g = f \circ h$ we have $g = h$;
- *epimorphism* if for any Z and any $g, h \in \operatorname{Hom}_{\mathscr{C}}(Y,Z)$ such that $g \circ f = h \circ f$ we have $g = h$;
- *split monomorphism* (also called *section*) if there is $g \in \operatorname{Hom}_{\mathscr{C}}(Y,X)$ such that $g \circ f = \mathrm{id}_X$;
- *split epimorphism* (also called *retraction*) if there is $g \in \operatorname{Hom}_{\mathscr{C}}(Y,X)$ such that $f \circ g = \mathrm{id}_Y$;

- *isomorphism* if there is $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

We often denote monomorphisms by arrows $\rightarrowtail$, and epimorphisms by arrows $\twoheadrightarrow$.

Exercise 2.9. Show that

- Any split monomorphism is a monomorphism.
- Any split epimorphism is an epimorphism.
- The following are equivalent, for a morphism f:
  - f is an isomorphism;
  - f is a split monomorphism and an epimorphism;
  - f is a monomorphism ans a split epimorphism.
- If f is an isomorphism, the the g in the definition above is uniquely determined. We denote it by f^{-1}.

Example 2.10.
- In **Set**: monomorphism are split monomorphisms are injective maps; epimorphisms are split epimorphisms are surjective maps; isomorphisms are bijective maps.
- For a poset $(X, \leqslant)$, all morphisms in the poset category $\mathscr{C}_{(X,\leqslant)}$ are both mono- and epimorphisms. However, only identities are split monomorphisms or split epimorphisms.

  In particular being a mono- and an epimorphism does not imply being an isomorphism.

3 Functors

Definition 3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $\mathtt{F}$ from $\mathscr{C}$ to $\mathscr{D}$ consists of

- a map $\mathcal{O}\mathfrak{b}\,\mathscr{C} \longrightarrow \mathcal{O}\mathfrak{b}\,\mathscr{D} \colon X \longmapsto \mathtt{F}X$, and
- for any $X, Y \in \mathcal{O}\mathfrak{b}\,\mathscr{C}$, a map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}X, \mathtt{F}Y)$, also denoted by $\mathtt{F}$,

such that

- for any $X \in \mathcal{O}b\,\mathscr{C}$ we have $\mathtt{F}\,\mathrm{id}_X = \mathrm{id}_{\mathtt{F}X}$, and
- for any composable morphisms f and g in $\mathscr{C}$ we have $\mathtt{F}(g \circ f) = \mathtt{F}g \circ \mathtt{F}f$.

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{D}$. In other words, it consists of maps $\mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}Y, \mathtt{F}X)$, and the composition rule is $\mathtt{F}(g \circ f) = \mathtt{F}f \circ \mathtt{F}g$.

Example 3.2.
- Let $\mathscr{S}$ be a subcategory of $\mathscr{C}$. Then inclusion $\mathscr{S} \hookrightarrow \mathscr{C}$ is a (covariant) functor.
- Let $\mathscr{C}$ be a category, and X be an object. Then $\mathrm{Hom}_{\mathscr{C}}(X,-)$ defines a functor from $\mathscr{C}$ to **Set**: For any object $Y \in \mathcal{O}b\,\mathscr{C}$, we obtain a set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ by definition of category. For a morphism $f \in \mathrm{Hom}_{\mathscr{C}}(Y_1, Y_2)$ we define $\mathrm{Hom}_{\mathscr{C}}(X,f)$ by

$$\mathrm{Hom}_{\mathscr{C}}(X,f)\colon\ \mathrm{Hom}_{\mathscr{C}}(X,Y_1) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,Y_2)\colon\ g \longmapsto f \circ g.$$

  This functor is called the *covariant* Hom-*functor*.
- Similarly one defines the *contravariant* Hom-*functor* $\mathrm{Hom}_{\mathscr{C}}(-,X)$.
- For two posets $(X, \leqslant)$ and $(Y, \leqslant)$, a functor between the poset categories is given by an order-preserving map $X \longrightarrow Y$.
- Forming fundamental groups gives a functor $\mathbf{Top}_* \longrightarrow \mathbf{Gp}$ from pointed topological spaces to groups.

Definition 3.3. A functor $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ is called

- *faithful* if for any $X, Y \in \mathcal{O}b\,\mathscr{C}$ the map $\mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}X, \mathtt{F}Y)$ is injective;
- *full* if for any $X, Y \in \mathcal{O}b\,\mathscr{C}$ the map $\mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}X, \mathtt{F}Y)$ is surjective;
- *dense* if for any $D \in \mathcal{O}b\,\mathscr{D}$ there is $C \in \mathcal{O}b\,\mathscr{C}$ such that $D \cong \mathtt{F}C$.

Example 3.4.
- For an order preserving map f between two posets X and Y, the associated functor between the poset categories is always faithful. It is full if the images of two points are only comparable in Y if the two points already were comparable in X. It is dense if and only if the map is surjective.
- The forgetful functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$ is faithful, but neither full nor dense.

Definition 3.5. Let $\mathscr{X}$ and $\mathscr{C}$ be categories. A *$\mathscr{C}$-valued presheaf on $\mathscr{X}$* is a functor

$$\mathscr{X}^{\mathrm{op}} \longrightarrow \mathscr{C}.$$

We denote by $\mathrm{presh}_{\mathscr{C}}\, \mathscr{X}$ the collection of all $\mathscr{C}$-valued presheaves on $\mathscr{X}$.

By abuse of notation, for a poset $(X, \leqslant)$, we say a presheaf on $(X, \leqslant)$ is a presheaf on the poset category $\mathscr{C}_{(X,\leqslant)}$.

More explicitly, a $\mathscr{C}$-valued presheaf F on a poset $(X, \leqslant)$ consist of

- for every $x \in X$, an object $F_x \in \mathcal{O}\mathbf{b}_{\mathscr{C}}$;
- for every $x, y \in X$, such that $x \leqslant y$, a morphism $\mathrm{res}_x^y \in \mathrm{Hom}_{\mathscr{C}}(F_y, F_x)$;

such that $\mathrm{res}_x^x = \mathrm{id}_{F_x}$, and $\mathrm{res}_x^y \circ \mathrm{res}_y^z = \mathrm{res}_x^z$, whenever $x \leqslant y \leqslant z$.

Remark 3.6. Depending on the setup, and preferences of different authors, various different notations are being used in the literature. These include $\mathrm{Fun}(\mathscr{X}^{\mathrm{op}}, \mathscr{C})$, and, in partticular in the case of posets, "representations of $\mathscr{X}$ in $\mathscr{C}$".

This name "presheaves" which we will use here originates in the following example.

Example 3.7. Let T be a topological space, and X the set of open subsets of T. Then X is a poset with inclusion as partial order. Let S be a set (possibly with some extra structure, for instance $S = \mathbb{R}$ or $S = \mathbb{C}$).

Then we obtain a **Set**-valued presheaf F on X by setting $F(U)$ to be all (nice) functions from U to S. Here the restriction morphisms are restriction of functions to a smaller open subset of T.

4 Natural transformations

Definition 4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $\mathtt{F}$ and $\mathtt{G}$ be functors from $\mathscr{C}$ to $\mathscr{D}$. A *natural transformation* η form $\mathtt{F}$ to $\mathtt{G}$ consists of

- for every $C \in \mathcal{O}\mathfrak{b}\,\mathscr{C}$ a morphism $\eta_C \in \operatorname{Hom}_{\mathscr{D}}(\mathtt{F}C, \mathtt{G}C)$,

such that for any morphism $f \in \operatorname{Hom}_{\mathscr{C}}(C_1, C_2)$ the we have

$$\eta_{C_2} \circ \mathtt{F}f = \mathtt{G}f \circ \eta_{C_1},$$

that is the following diagram commutes in $\mathscr{D}$:

$$\begin{tikzcd} \mathtt{F}C_1 \arrow[r, "\eta_{C_1}"] \arrow[d, "\mathtt{F}f"'] & \mathtt{G}C_1 \arrow[d, "\mathtt{G}f"] \\ \mathtt{F}C_2 \arrow[r, "\eta_{C_2}"] & \mathtt{G}C_2 \end{tikzcd}$$

A natural transformation η is called *natural isomorphism* if all the η_C are isomorphisms in $\mathscr{D}$.

Example 4.2. Let R be a ring. We denote by $-^* = \operatorname{Hom}_R(-, R)$ the duality $\operatorname{Mod} R \longrightarrow \operatorname{Mod} R^{\mathrm{op}}$ with respect to the ring. Then $-^{**}$ defines a functor $\operatorname{Mod} R \longrightarrow \operatorname{Mod} R$, and we have a natural transformation given by evaluation:

$$\begin{aligned} &\mathrm{ev}\colon \mathrm{id}_{\operatorname{Mod} R} \longrightarrow -^{**} \\ &\mathrm{ev}_M(m) = [M^* \ni \phi \longmapsto \phi(m) \in R] \in \operatorname{Hom}_R(M^*, R). \end{aligned}$$

For $R = \mathbb{F}$ a field, we note that $-^{**}$ also defines a functor $\operatorname{mod} \mathbb{F} \longrightarrow \operatorname{mod} \mathbb{F}$ between the categories of finite dimensional modules, and the induced natural transformation

$$\mathrm{ev}\colon \mathrm{id}_{\operatorname{mod} \mathbb{F}} \longrightarrow -^{**}$$

is a natural isomorphism.

Observation 4.3. Let $\mathscr{X}$ be a category, such that the objects form a set. (Such a category is called *small*.)

Then, for an arbitrary category $\mathscr{C}$ and functors $\mathtt{F}, \mathtt{G}\colon \mathscr{X} \longrightarrow \mathscr{C}$ the collection of natural transformations from $\mathtt{F}$ to $\mathtt{G}$ forms a set. (In fact, it is a subset of $\times_{X \in \mathcal{O}\mathfrak{b}\,\mathscr{X}} \operatorname{Hom}_{\mathscr{C}}(\mathtt{F}X, \mathtt{G}X)$.)

Thus, for a small category $\mathscr{X}$, the $\mathscr{C}$-valued presheaves on $\mathscr{X}$ form a category, with

$$\operatorname{Hom}_{\operatorname{presh}_{\mathscr{C}} \mathscr{X}}(F_1, F_2) = \{\text{natural transformations } F_1 \longrightarrow F_2\}.$$

Obviously natural isomorphisms are precisely the isomorphisms in $\operatorname{presh}_{\mathscr{C}} \mathscr{X}$.

Example 4.4. Let $(X, \leqslant)$ be a poset, $\mathscr{C}$ a category, and F_1 and F_2 $\mathscr{C}$-valued presheaves on X.

A morphism $f\colon F_1 \longrightarrow F_2$ consists of morphisms $f_x\colon (F_1)_x \longrightarrow (F_2)_x$ for any $x \in X$, such that $\mathrm{res}^y_x \circ f_y = f_x \circ \mathrm{res}^y_x$ whenever $x \leqslant y$. (Note that here the left restriction refers to the structure of F_2, while the right restriction comes from the structure of F_1.)

Example 4.5.

- Let $X = \{1\}$ be the poset with just one element. Then $\mathrm{presh}_{\mathscr{C}} X = \mathscr{C}$.
- Let $X = \{1 \leqslant 2\}$ be the poset with two comparable elements. Then the objects of $\mathrm{presh}_{\mathscr{C}} X$ are morphisms in $\mathscr{C}$, and morphisms of presheaves are pairs of morphisms between domains and codomains, such that the resulting square commutes.
- Let X be the poset given by the Hasse diagram

$$\begin{tikzcd}[column sep=small, row sep=small] & \omega \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

The objects of $\mathrm{presh}_{\mathscr{C}} X$ are commutative squares in $\mathscr{C}$. (Note that we don't need to specify res^ω_0, since $\mathrm{res}^\omega_0 = \mathrm{res}^a_0 \circ \mathrm{res}^\omega_a = \mathrm{res}^b_0 \circ \mathrm{res}^{\omega_b}$.)

Theorem 4.6 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category, $C \in \mathfrak{Ob}\,\mathscr{C}$, and $\mathtt{F}$ a functor $\mathscr{C} \longrightarrow \mathbf{Set}$. Then the map*

$$Y\colon \{\textit{natural transformations } \mathrm{Hom}_{\mathscr{C}}(C, -) \longrightarrow \mathtt{F}\} \longrightarrow \mathtt{F}C$$
$$\eta \longmapsto \eta_C(\mathrm{id}_C)$$

is a bijection. In particular the natural transformations from $\mathrm{Hom}_{\mathscr{C}}(C, -)$ *to* $\mathtt{F}$ *form a set.*

Proof. We construct a map in the opposite direction. That is, given an element $x \in \mathtt{F}C$, we construct a natural transformation $\zeta^x\colon \mathrm{Hom}_{\mathscr{C}}(C, -) \longrightarrow \mathtt{F}$. For $D \in \mathfrak{Ob}\,\mathscr{C}$ we set

$$\zeta^x_D\colon \mathrm{Hom}_{\mathscr{C}}(C, D) \longrightarrow \mathtt{F}D\colon f \longmapsto (\mathtt{F}f)(x).$$

(Note that $\mathtt{F}f \in \mathrm{Hom}_{\mathbf{Set}}(\mathtt{F}C, \mathtt{F}D)$, so this makes sense.)

Let us first check that ζ^x is a natural transformation. Let $g \in \mathrm{Hom}_{\mathscr{C}}(D_1, \mathsf{D}_2)$. We have

$$\begin{aligned}\zeta^x_{D_2} \circ \mathrm{Hom}_{\mathscr{C}}(C, g) &= [f \longmapsto (\mathtt{F}f)(x)] \circ [f \longmapsto g \circ f] \\ &= [f \longmapsto (\mathtt{F}(g \circ f))(x)] \\ &= \mathtt{F}(g) \circ [f \longmapsto (\mathtt{F}f)(x)] \\ &= \mathtt{F}(g) \circ \zeta^x_{D_1}\end{aligned}$$

We immediately see that

$$Y(\zeta^x) = \zeta^x_C(\mathrm{id}_C) = (\mathtt{F}\,\mathrm{id}_C)(x) = \mathrm{id}_{\mathtt{F}C}\, x = x.$$

It remains to see that for any natural transformation $\eta\colon \mathrm{Hom}_{\mathscr{C}}(C, -) \longrightarrow \mathtt{F}$ we have $\eta = \zeta^{Y(\eta)}$. So Let $D \in \mathcal{O}\mathfrak{b}\,\mathscr{C}$. Then

$$\begin{aligned}\zeta^{Y(\eta)}_D &= [f \longmapsto (\mathtt{F}f)(Y\eta)] \\ &= [f \longmapsto (\mathtt{F}f \circ \eta_C)(\mathrm{id}_C)] \\ &= [f \longmapsto (\eta_D \circ \mathrm{Hom}_{\mathscr{C}}(C, f))(\mathrm{id}_C)] \quad (\eta \text{ is a natural transformation}) \\ &= [f \longmapsto \eta_D(f)] \\ &= \eta_D.\end{aligned}$$

□

Corollary 4.7 (Yoneda embedding)**.** *Let $\mathscr{X}$ be a small category. Then the functor*

$$\mathtt{Y}\colon \mathscr{X} \longrightarrow \mathrm{presh}_{\mathbf{Set}}\,\mathscr{X} : X \longmapsto \mathrm{Hom}_{\mathscr{X}}(-, X)$$

is fully faithful.

5 Equivalences of categories

Definition 5.1. A functor $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ is called an *equivalence* if there is a functor $\mathtt{G}\colon \mathscr{D} \longrightarrow \mathscr{C}$ such that $\mathtt{F} \circ \mathtt{G} \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{D}}$ and $\mathtt{G} \circ \mathtt{F} \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{C}}$.

Theorem 5.2. *A functor $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ is an equivalence if and only if it is full, faithful, and dense.*

Proof. Assume first that $\mathtt{F}$ is an equivalence, and let $\mathtt{G}$ as in the definition.

Let $\eta\colon \mathtt{G}\circ\mathtt{F}\longrightarrow \mathrm{id}_{\mathscr{C}}$ be a natural isomorphism. Then for any morphism $f\in\mathrm{Hom}_{\mathscr{C}}(C_1,C_2)$ we have the commutative square

$$\begin{tikzcd}
\mathtt{GF}C_1 \arrow[r, "\eta_{C_1}"] \arrow[d, "\mathtt{GF}f"'] & C_1 \arrow[d, "f"] \\
\mathtt{GF}C_2 \arrow[r, "\eta_{C_2}"] & C_2
\end{tikzcd}$$

Thus $f=\eta_{C_2}\circ\mathtt{GF}f\circ\eta_{C_1}^{-1}$ is uniquely determined by $\mathtt{F}f$. That is $\mathtt{F}$ is faithful.

Let ζ ba a natural isomorphism $\mathtt{F}\circ\mathtt{G}\longrightarrow\mathrm{id}_{\mathscr{D}}$. In particular for any $D\in\mathcal{O}\mathfrak{b}\,\mathscr{D}$ we have an isomorphism $\zeta_D\colon\mathtt{FG}D\longrightarrow D$, showing that $\mathtt{F}$ is dense.

To see that $\mathtt{F}$ is full, let $f\in\mathrm{Hom}_{\mathscr{D}}(\mathtt{F}C_1,\mathtt{F}C_2)$. Using the natural isomorphisms η and ζ as above, we construct the commutative diagram

$$\begin{tikzcd}
\mathtt{F}C_1 \arrow[d, "f"'] & \mathtt{FGF}C_1 \arrow[l, "\mathtt{F}\eta_{C_1}"'] \arrow[r, "\zeta_{\mathtt{F}C_1}"] \arrow[d, "g"'] & \mathtt{F}C_1 \arrow[d, "h"'] \\
\mathtt{F}C_2 & \mathtt{FGF}C_2 \arrow[l, "\mathtt{F}\eta_{C_2}"'] \arrow[r, "\zeta_{\mathtt{F}C_2}"] & \mathtt{F}C_2
\end{tikzcd}$$

where g and h are the unique maps making the squares commutative. By naturality of ζ we know that $g=\mathtt{FG}h$, and thus the commutativity of the left hand square gives us that

$$\begin{aligned}
f &= \mathtt{F}\eta_{C_2}\circ\mathtt{FG}h\circ(\mathtt{F}\eta_{C_1})^{-1}\\
&= \mathtt{F}(\eta_{C_2}\circ\mathtt{G}h\circ\eta_{C_1}^{-1})
\end{aligned}$$

showing that f is in the image of $\mathtt{F}$.

Now assume conversely that $\mathtt{F}$ is full, faithful, and dense. By (a strong version of) the axiom of choice, and since $\mathtt{F}$ is dense, we may fix, for any $D\in\mathcal{O}\mathfrak{b}_{\mathscr{D}}$, an object $\mathtt{G}D$ in $\mathscr{C}$ and an isomorphism $\zeta_D\colon\mathtt{FG}D\longrightarrow D$. For a morphism $f\in\mathrm{Hom}_{\mathscr{D}}(D_1,D_2)$ we use the bijection

$$\mathrm{Hom}_{\mathscr{C}}(\mathtt{G}D_1,\mathtt{G}D_2)\longrightarrow\mathrm{Hom}_{\mathscr{D}}(\mathtt{FG}D_1,\mathtt{FG}D_2)$$

induced by $\mathtt{F}$ (since it is full and faithful), and define $\mathtt{G}f$ to be the preimage of $\zeta_{D_2}^{-1} \circ f \circ \zeta_{D_1}$.

We claim that the above makes $\mathtt{G}$ a functor from $\mathscr{D}$ to $\mathscr{C}$. Firstly we have

$$\mathtt{G}\,\mathrm{id}_D = \mathtt{F}^{-1}(\zeta_D^{-1} \circ \mathrm{id}_D \circ \zeta_D) = \mathtt{F}^{-1}(\mathrm{id}_{\mathtt{FG}D}) = \mathrm{id}_{\mathtt{G}D}\,.$$

Secondly, for morphisms $D_1 \xrightarrow{f} D_2 \xrightarrow{g} D_3$,

$$\begin{aligned}\mathtt{G}(g \circ f) &= \mathtt{F}^{-1}(\zeta_{D_3}^{-1} \circ g \circ f \circ \zeta_{D_1}) = \mathtt{F}^{-1}(\zeta_{D_3}^{-1} \circ g \circ \zeta_{D_2} \circ \zeta_{D_2}^{-1} f \circ \zeta_{D_1}) \\ &= \mathtt{F}^{-1}(\zeta_{D_3}^{-1} \circ g \circ \zeta_{D_2}) \circ \mathtt{F}^{-1}(\zeta_{D_2}^{-1} f \circ \zeta_{D_1}) = \mathtt{G}g \circ \mathtt{G}f.\end{aligned}$$

Next we claim that ζ defines a natural isomorphism $\mathtt{F} \circ \mathtt{G} \longrightarrow \mathrm{id}_{\mathscr{D}}$. Let $f \in \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$. Then

$$\zeta_{D_2} \circ \mathtt{FG}f = \zeta_{D_2} \circ \zeta D_2{}^{-1} \circ f \circ \zeta_{D_1} = f \circ \zeta_{D_1}.$$

Finally, we construct a natural isomorphism $\eta\colon \mathtt{G} \circ \mathtt{F} \longrightarrow \mathrm{id}_{\mathscr{C}}$. First note that ζ induces mutually inverse natural isomorphisms

$$\zeta_{\mathtt{F}-}\colon \mathtt{F} \circ \mathtt{G} \circ \mathtt{F} \longrightarrow \mathtt{F} \text{ and } \zeta_{\mathtt{F}-}^{-1}\colon \mathtt{F} \longrightarrow \mathtt{F} \circ \mathtt{G} \circ \mathtt{F}.$$

Since $\mathtt{F}$ is fully faithful, we can find unique morphisms $\eta_C\colon \mathtt{GF}C \longrightarrow C$ and $\eta_C^-\colon C \longrightarrow \mathtt{GF}C$ such that

$$\zeta_{\mathtt{F}C} = F\eta_C \text{ and } \zeta_{\mathtt{F}C}^{-1} = F\eta_C^-.$$

If follows that η is a natural transformation, with inverse η^-. □

Example 5.3. Let $\mathbb{F}$ be a field.

Let $\mathbf{Mat}_{\mathbb{F}}$ be the category given by

$$\begin{aligned}&\mathcal{Ob}\,\mathbf{Mat}_{\mathbb{F}} = \mathbb{N}_0, \text{ and} \\ &\mathrm{Hom}_{\mathbf{Mat}_{\mathbb{F}}}(m, n) = \{n \times m\text{-matrices over } \mathbb{F}\}\end{aligned}$$

with matrix multiplication.

Let $\mathrm{mod}\,\mathbb{F}$ be the category of finite dimensional $\mathbb{F}$-vector spaces, with $\mathbb{F}$-vector space homomorphisms as morphisms.

Then the natural functor $\mathbf{Mat}_{\mathbb{F}} \longrightarrow \mathrm{mod}\,\mathbb{F}$ sending n to $\mathbb{F}^n$ is an equivalence.

We observe that constructing an equivalence in the other direction amounts to choosing a basis for every finite dimensional $\mathbb{F}$-vector space.

6 Adjoint functors

Definition 6.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathtt{G}\colon \mathscr{D} \longrightarrow \mathscr{C}$ functors. We say that $(\mathtt{F}, \mathtt{G})$ is an *adjoint pair* if the functors

$$\mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-, -) \text{ and } \mathrm{Hom}_{\mathscr{C}}(-, \mathtt{G}-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \longrightarrow \mathbf{Set}$$

are naturally isomorphic.

Example 6.2. Let $(X, \leqslant)$ be a poset, and $\mathscr{C}$ a category. For $x \in X$ we have a natural projection functor

$$\pi_x\colon \operatorname{presh}_{\mathscr{C}} X \longrightarrow \mathscr{C}\colon F \longmapsto F_x.$$

We may also consider the diagonal functor $\Delta\colon \mathscr{C} \longrightarrow \operatorname{presh}_{\mathscr{C}} X$ given by $\Delta(C)_x = C$ for any $x \in X$, and $\operatorname{res}^x_y = \mathrm{id}_C$ for any $y \leqslant x$.

If $(X, \leqslant)$ has a smallest element 0, then (π_0, Δ) is an adjoint pair. Similarly, if there is a largest element ω, then (Δ, π_ω) is an adjoint pair.

Example 6.3 (Free modules)**.** Let R be a ring. Then we have the forgetful functor $\mathtt{f}\colon \operatorname{Mod} R \longrightarrow \mathbf{Set}$.

We construct a left adjoint $R^{(-)}\colon \mathbf{Set} \longrightarrow \operatorname{Mod} R$: For a set X,

$$R^{(X)} = \{\text{functions } f\colon X \longrightarrow R \mid f(x) \neq 0 \text{ for only finitely many } x \in X\}.$$

For a map $\varphi\colon X \longrightarrow Y$ we set

$$R^{(f)}\colon R^{(X)} \longrightarrow R^{(Y)}\colon f \longmapsto \left[y \longmapsto \sum_{x \in \varphi^{-1}(y)} f(x) \right].$$

We claim that $R^{(-)}$ is left adjoint to $\mathtt{f}$.

For $x \in X$, we let

$$\chi_x\colon X \longrightarrow R\colon y \longmapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}.$$

Then $\chi_x \in R^{(X)}$.

Now we can define the mutually inverse natural transformations by

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Set}}(X, \mathtt{f}M) &\longleftrightarrow \mathrm{Hom}_R(R^{(X)}, M) \\
\varphi &\longmapsto [f \longmapsto \sum_{x \in X} \varphi(x) \cdot f(x)] \\
[x \longmapsto \psi(\chi_x)] &\longleftarrow \psi
\end{aligned}$$

(Note that the sum in the second line is finite, since $f(x) = 0$ for almost all $x \in X$.)

Example 6.4. Consider the forgetful functor $\mathtt{f}\colon \mathbf{Ab} \longrightarrow \mathbf{Gp}$. This functor has a left adjoint, given by forming commutator factor groups.

Proposition 6.5 (Unit-counit adjunction)**.** *Let* $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ *and* $\mathtt{G}\colon \mathscr{D} \longrightarrow \mathscr{C}$ *be two functors. Then the following are equivalent:*

1. $(\mathtt{F}, \mathtt{G})$ *is an adjoint pair.*
2. *there are natural transformations* $\eta\colon \mathrm{id}_{\mathscr{C}} \longrightarrow \mathtt{G} \circ \mathtt{F}$ *and* $\varepsilon\colon \mathtt{F} \circ \mathtt{G} \longrightarrow \mathrm{id}_{\mathscr{D}}$ *(called* unit *and* counit*, respectively), such that*

$$\mathrm{id}_{\mathtt{F}} = \varepsilon_{\mathtt{F}-} \circ \mathtt{F}\eta \text{ and } \mathrm{id}_{\mathtt{G}} = \mathtt{G}\varepsilon \circ \eta_{\mathtt{G}-},$$

i.e. for any $C \in \mathcal{O}\mathrm{b}\, \mathscr{C}$ *and* $D \in \mathcal{O}\mathrm{b}\, \mathscr{D}$

$$\mathrm{id}_{\mathtt{F}C} = \varepsilon_{\mathtt{F}C} \circ \mathtt{F}\eta_C \text{ and } \mathrm{id}_{\mathtt{G}D} = \mathtt{G}\varepsilon_D \circ \eta_{\mathtt{G}D}.$$

Proof. Let $\alpha\colon \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-, -) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(-, \mathtt{G}-)$ be a natural transformation.

Then we may define a natural transformation $\eta\colon \mathrm{id}_{\mathscr{C}} \longrightarrow \mathtt{G} \circ \mathtt{F}$ by

$$\eta_C = \alpha_{C,\mathtt{F}C}(\mathrm{id}_{\mathtt{F}C}).$$

To check that this defines a natural transformation, note that for a morphism $f \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_2)$ we have

$$\begin{aligned}
\mathtt{GF}f \circ \eta_{C_1} &= \mathtt{GF}f \circ \alpha_{C_1,\mathtt{F}C_1}(\mathrm{id}_{\mathtt{F}C_1}) \\
&= \alpha_{C_1,\mathtt{F}C_2}(\mathtt{F}f) \\
&= \alpha_{C_2,\mathtt{F}C_2}(\mathrm{id}_{\mathtt{F}C_2}) \circ f \\
&= \eta_{C_2} \circ f,
\end{aligned}$$

where the middle two equalities follow from the naturality of α in the second and first argument, respectively.

Conversely, given a natural transformation $\eta\colon \mathrm{id}_{\mathscr{C}} \longrightarrow \mathtt{G} \circ \mathtt{F}$ we can define a natural transformation $\alpha\colon \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-,-) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(-,\mathtt{G}-)$ by

$$\alpha_{C,D}(f) = \mathtt{G}(f) \circ \eta_C.$$

It is immediately checked that these two constructions are mutually inverse.

Similarly, we obtain a bijection between natural transformations

$$\beta\colon \mathrm{Hom}_{\mathscr{C}}(-,\mathtt{G}-) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-,-) \text{ and } \varepsilon\colon \mathtt{F} \circ \mathtt{G} \longrightarrow \mathrm{id}_{\mathscr{D}},$$

sending β to the natural transformation given by $\varepsilon_D = \beta_{\mathtt{G}D,D}(\mathrm{id}_{\mathtt{G}D})$.

Now let α and η and β and ε correspond to each other as above. Then

$$\begin{aligned} &\beta \circ \alpha = \mathrm{id}_{\mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-,-)} \\ \iff &\forall C \in \mathfrak{Ob}\,\mathscr{C}\ \forall D \in \mathfrak{Ob}\,\mathscr{D}\colon \beta_{C,D} \circ \alpha_{C,D} = \mathrm{id}_{\mathrm{Hom}_{\mathscr{D}}(\mathtt{F}C,D)} \end{aligned}$$

Moreover, since any morphism from $\mathtt{F}C$ to D is a multiple of $\mathrm{id}_{\mathtt{F}C}$

$$\iff \forall C \in \mathfrak{Ob}\,\mathscr{C}\colon \beta_{C,\mathtt{F}C} \circ \alpha_{C,\mathtt{F}C}(\mathrm{id}_{\mathtt{F}C}) = \mathrm{id}_{\mathtt{F}C}$$

and, inserting $\alpha_{C,\mathtt{F}C}(\mathrm{id}_{\mathtt{F}C}) = \mathtt{G}(\mathrm{id}_{\mathtt{F}C}) \circ \eta_C = \eta_C$, and $\beta_{C,\mathtt{F}C}(\eta_C) = \varepsilon_{\mathtt{F}C} \circ \mathtt{F}(\eta_C)$, we obtain

$$\begin{aligned} \iff &\forall C \in \mathfrak{Ob}\,\mathscr{C}\colon \varepsilon_{\mathtt{F}C} \circ \mathtt{F}(\eta_C) = \mathrm{id}_{\mathtt{F}C} \\ \iff &\varepsilon_{\mathtt{F}-} \circ \mathtt{F}\eta = \mathrm{id}_{\mathtt{F}}\,. \end{aligned}$$

Similarly one can see that $\alpha \circ \beta = \mathrm{id}_{\mathrm{Hom}_{\mathscr{C}}(-,\mathtt{G}-)}$ if and only if $\mathtt{G}\varepsilon \circ \eta_{\mathtt{G}-} = \mathrm{id}_{\mathtt{G}}$. □

7 Limits

Definition 7.1. Let $\mathscr{X}$ be a small category (which we think of as indices, in some sense), and $\mathscr{C}$ an arbitrary category. We denote by Δ the functor

$$\Delta\colon \mathscr{C} \longrightarrow \mathrm{presh}_{\mathscr{C}}\,\mathscr{X}\colon C \longmapsto \Delta C,$$

where ΔC is the functor sending any object of $\mathscr{X}$ to C, and any morphism to id_C.

Let $F \in \mathfrak{Ob}\,\mathrm{presh}_{\mathscr{C}}\,\mathscr{X}$.

- a *limit* (or inverse limit, projective limit) of F is an object $\varprojlim F \in \mathcal{O}\!\mathit{b}\, \mathscr{C}$, together with a natural isomorphism
$$\mathrm{Hom}_{\mathscr{C}}(-, \varprojlim F) \cong \mathrm{Hom}_{\mathrm{presh}_{\mathscr{C}}\, \mathscr{X}}(\Delta -, F)$$
of functors $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$;
- a *colimit* (or direct limit, injective limit) of F is an object $\varinjlim F \in \mathcal{O}\!\mathit{b}\, \mathscr{C}$, together with a natural isomorphism
$$\mathrm{Hom}_{\mathscr{C}}(\varinjlim F, -) \cong \mathrm{Hom}_{\mathrm{presh}_{\mathscr{C}}\, \mathscr{X}}(F, \Delta -)$$
of functors $\mathscr{C} \longrightarrow \mathbf{Set}$.

Observation 7.2. Note that a limit of F can equivalently be characterized as an object $\varprojlim F \in \mathcal{O}\!\mathit{b}\, \mathscr{C}$, together with morphisms $\varphi_x \colon \varprojlim F \longrightarrow F_x$ such that $f \circ \varphi_x = \varphi_y$ for any $f \colon x \longrightarrow y \in \mathscr{X}$, which is universal in the following sense: for any other object C, together with maps $\psi_x \colon C \longrightarrow F_x$ such that $f \circ \psi_x = \psi_y$ for any $f \colon x \longrightarrow y \in \mathscr{X}$ there is a unique map $\Psi \colon C \longrightarrow \varprojlim F$ such that $\psi_x = \varphi_x \circ \Psi$ for all $x \in \mathcal{O}\!\mathit{b}\, \mathscr{X}$.

The dual description holds for colimits.

Proposition 7.3. *Let $F \in \mathcal{O}\!\mathit{b}\, \mathrm{presh}_{\mathscr{C}}\, \mathscr{X}$ as above. If a limit $\varprojlim F$ exists, then it is unique up to (unique) isomorphism. If a colimit $\varinjlim F$ exists, then it is unique up to (unique) isomorphism.*

Therefore it makes sense to speak about the *limit or colimit.*

Proof. Let (L, φ_x) and (L', φ'_x) be two limits. Then, by the universal property for L, there is a morphism $\Psi \colon L' \longrightarrow L$ such that $\varphi'_x = \varphi_x \circ \Psi$. By the universal property for L' there is a morphism $\Psi \colon L \longrightarrow L'$ such that $\varphi_x = \varphi'_x \circ \Psi'$.

Now, again by the universal property of L, there exists a *unique* morphism $\Phi \colon L \longrightarrow L$ such that $\varphi_x = \varphi_x \circ \Phi$. But we know two candidates for Φ: id_L and $\Psi \circ \Psi'$. It follows that $\Psi \circ \Psi' = \mathrm{id}_L$. Similarly one sees that $\Psi' \circ \Psi = \mathrm{id}_{L'}$. It follows that Ψ and Ψ' are mutually inverse isomorphisms. $\square$

Observation 7.4. If any $F \in \mathrm{presh}_{\mathscr{C}}\, \mathscr{X}$ has a limit, then $\varprojlim$ defines a functor $\mathrm{presh}_{\mathscr{C}}\, \mathscr{X} \longrightarrow \mathscr{C}$, and this functor is right adjoint to Δ.

If any $F \in \mathrm{presh}_{\mathscr{C}}\, \mathscr{X}$ has a colimit, then $\varinjlim$ gives a functor $\mathrm{presh}_{\mathscr{C}}\, \mathscr{X} \longrightarrow \mathscr{C}$, and this functor is left adjoint to Δ.

Example 7.5. Let $(X, \leqslant)$ be a poset with a smallest element 0. Then we have seen in Example 6.2 that

$$\varinjlim = \pi_0 \colon \operatorname{presh}_{\mathscr{C}} X \longrightarrow \mathscr{C} \colon F \longmapsto F_0.$$

Similarly, if X has a largest element ω, then $\varprojlim = \pi_\omega$.

Definition 7.6 (Product and coproduct)**.** Let X be a set. We may regard X as a poset with trivial poset structure. Let $\mathscr{C}$ be a category, and $F \in \operatorname{presh}_{\mathscr{C}} X$. (That is F is a collection of objects F_x, one for each $x \in X$.)

- If the limit $\varprojlim F$ exists, then it is called *product* of the objects F_x, and denoted by $\prod_{x \in X} F_x$.
- If the colimit $\varinjlim F$ exists, then it is called *coproduct* of the objects F_x, and denoted by $\coprod_{x \in X} F_x$.

Example 7.7. In the category **Set**, products are cartesian products, and coproducts are disjoint unions.

Example 7.8. In $\operatorname{Mod} R$, both finite products and finite coproducts are given by direct sums.

Definition 7.9 (Pullback and pushout)**.** Let X be the poset given by the Hasse diagram

$$\begin{tikzcd}[column sep=small, row sep=small] a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

Let $F \in \operatorname{presh}_{\mathscr{C}} X$. If the limit $\varprojlim F$ exists, then it is called the *pullback* (or fibre product) of F, and denoted by $F_a \prod\limits_{F_0} F_b$.

By abuse of notation we also call the commutative square

$$\begin{tikzcd} F_\omega \arrow[r] \arrow[d] & F_a \arrow[d] \\ F_b \arrow[r] & F_0 \end{tikzcd}$$

a pullback, provided F_ω is the pullback of the rest of the diagram.

More explicitly, a pullback is a commutative square as above, such that for any other X with morphisms $X \longrightarrow F_a$ and $X \longrightarrow F_b$ making a similar square commutative, there is a unique morphism $X \longrightarrow F_\omega$ making the two triangles in the following diagram commutative.

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& F_\omega \arrow[r] \arrow[d] & F_a \arrow[d] \\
& F_b \arrow[r] & F_0
\end{tikzcd}$$

Let Y be the poset given by the Hasse diagram

$$\begin{tikzcd}[column sep=small, row sep=small]
& \omega \arrow[dl, no head] \arrow[dr, no head] & \\
a & & b
\end{tikzcd}$$

Let $F \in \mathrm{presh}_{\mathscr{C}}\, X$. If the colimit $\varinjlim F$ exists, then we call it the *pushout* of F, and denote it by $F_a \coprod\limits_{F_\omega} F_b$.

Example 7.10. In the category **Set**, the pullback of

$$\begin{tikzcd}
F_a \arrow[dr] & & F_b \arrow[dl] \\
& F_0 &
\end{tikzcd}$$

is given by

$$F_a \prod_{F_0} F_b = \{(a, b) \in F_a \times F_b \mid \mathrm{res}^a_0(a) = \mathrm{res}^b_0(b)\}.$$

The pushout of

$$\begin{tikzcd}
& F_\omega \arrow[dl] \arrow[dr] & \\
F_a & & F_b
\end{tikzcd}$$

is given by

$$F_a \coprod_{F_\omega} F_b = F_a \coprod F_b / (\mathrm{res}^\omega_a(x) \sim \mathrm{res}^\omega_b(x) \mid x \in F_\omega).$$

8 Limits and adjoint functors

Construction 8.1. Let $\mathtt{F}\colon \mathscr{C} \longrightarrow \mathscr{D}$ be a functor, and $\mathscr{X}$ be a small category. Then $\mathtt{F}$ induces a functor $\mathtt{F}^{\text{presh}}\colon \operatorname{presh}_{\mathscr{C}} \mathscr{X} \longrightarrow \operatorname{presh}_{\mathscr{D}} \mathscr{X}$.

Lemma 8.2. *Let $(\mathtt{F}, \mathtt{G})$ be an adjoint pair of functors between categories $\mathscr{C}$ and $\mathscr{D}$. Let $\mathscr{X}$ be a small category.*

Then the corresponding functors between presheaf categories

$$\mathtt{F}^{\text{presh}}\colon \operatorname{presh}_{\mathscr{C}} \mathscr{X} \longrightarrow \operatorname{presh}_{\mathscr{D}} \mathscr{X} \text{ and } \mathtt{G}^{\text{presh}}\colon \operatorname{presh}_{\mathscr{D}} \mathscr{X} \longrightarrow \operatorname{presh}_{\mathscr{C}} \mathscr{X}$$

also form an adjoint pair.

Proof. We have a natural isomorphism $\eta\colon \operatorname{Hom}_{\mathscr{D}}(\mathtt{F}-, -) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(-, \mathtt{G}-)$, i.e. a collection of bijections $\eta_{X,Y}\colon \operatorname{Hom}_{\mathscr{D}}(\mathtt{F}X, Y) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X, \mathtt{G}Y)$ such that

- for any $f\colon X \longrightarrow X' \in \mathscr{C}$ we have a commutative square

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{D}}(\mathtt{F}X, Y) \arrow[r, "\eta_{X,Y}"] & \operatorname{Hom}_{\mathscr{D}}(X, \mathtt{G}Y) \\
\operatorname{Hom}_{\mathscr{D}}(\mathtt{F}X', Y) \arrow[u, "\operatorname{Hom}_{\mathscr{D}}(\mathtt{F}f, Y)"] \arrow[r, "\eta_{X',Y}"] & \operatorname{Hom}_{\mathscr{D}}(X', \mathtt{G}Y) \arrow[u, "\operatorname{Hom}_{\mathscr{C}}(f, \mathtt{G}Y)"']
\end{tikzcd}$$

  that is for any $\varphi\colon \mathtt{F}X' \longrightarrow Y \in \mathscr{D}$ we have

$$\eta_{X,Y}(\varphi \circ \mathtt{F}f) = \eta_{X',Y}(\varphi) \circ f.$$

- similarly, for any $g\colon Y \longrightarrow Y' \in \mathscr{D}$ and any $\varphi \in \operatorname{Hom}_{\mathscr{D}}(X, Y)$ we have

$$\mathtt{G}g \circ \eta_{X,Y}(\varphi) = \eta_{X,Y'}(g \circ \varphi).$$

Now we observe that

$$\begin{aligned}
&\operatorname{Hom}_{\operatorname{presh}_{\mathscr{D}} \mathscr{X}}(\mathtt{F}^{\text{presh}} S, T) \\
=&\{(f_x)_{x \in \mathfrak{Ob}\, \mathscr{X}} \in \bigtimes_{x \in \mathfrak{Ob}\, \mathscr{X}} \operatorname{Hom}_{\mathscr{D}}(\mathtt{F}S_x, T_x) \mid \forall \alpha \in \operatorname{Hom}_{\mathscr{X}}(x, y)\colon f_x \circ \mathtt{F}S_\alpha = T_\alpha \circ f_y\}
\end{aligned}$$

Now $f_x \circ S_\alpha$ and $T_\alpha \circ f_y$ are morphisms from $\mathtt{F}S_y$ to T_x. Since η_{S_y,T_X} is a bijection we may replace the conditions above by $\eta_{S_y,T_X}(f_x \circ \mathtt{F}S_\alpha) = \eta_{S_y,T_x}(T_\alpha \circ f_y)$.

Now note that by the two bullet points above the left hand side is equal to $\eta_{S_x,T_x}(f_x) \circ S_\alpha$, while the right hand side is equal to $\mathtt{G}T_\alpha \eta_{S_y,T_y}(f_y)$. Thus, writing g_i for $\eta_{S_x,T_x}(f_x)$, the above set is in bijection to

$$\{(g_x)_{x \in \mathfrak{Ob}\,\mathscr{X}} \in \bigtimes_{x \in \mathfrak{Ob}\,\mathscr{X}} \mathrm{Hom}_{\mathscr{C}}(S_x, \mathtt{G}T_x) \mid \forall \alpha \in \mathrm{Hom}_{\mathscr{X}}(x,y) \colon g_x \circ S_\alpha = \mathtt{G}T_\alpha \circ g_y\}$$
$$= \mathrm{Hom}_{\mathrm{presh}_{\mathscr{C}}\,\mathscr{X}}(S, \mathtt{G}^{\mathrm{presh}} T)$$

$\square$

Theorem 8.3. *Let* $(\mathtt{F}, \mathtt{G})$ *be an adjoint pair of functors between categories* $\mathscr{C}$ *and* $\mathscr{D}$. *Let* $\mathscr{X}$ *be a small category.*

- *Let* $X \in \mathrm{presh}_{\mathscr{D}}\,\mathscr{X}$ *such that* $\varprojlim X$ *exists. Then*
$$\varprojlim \mathtt{G}^{\mathrm{presh}} X = \mathtt{G} \varprojlim X.$$
(In particular this limit also exists.)
- *Let* $X \in \mathrm{presh}_{\mathscr{C}}\,\mathscr{X}$ *such that* $\varinjlim X$ *exists. Then*
$$\varinjlim \mathtt{F}^{\mathrm{presh}} X = \mathtt{F} \varinjlim X.$$

Motto: Right adjoints commute with limits, left adjoints commute with colimits.

Proof. We only prove the first claim, the second one is dual.

We have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(-, \mathtt{G} \varprojlim X) &\cong \mathrm{Hom}_{\mathscr{D}}(\mathtt{F}-, \varprojlim X) \\
&\cong \mathrm{Hom}_{\mathrm{presh}_{\mathscr{D}}\,\mathscr{X}}(\Delta \mathtt{F}-, X) \\
&= \mathrm{Hom}_{\mathrm{presh}_{\mathscr{D}}\,\mathscr{X}}(\mathtt{F}^{\mathrm{presh}} \Delta -, X) \\
&= \mathrm{Hom}_{\mathrm{presh}_{\mathscr{C}}\,\mathscr{X}}(\Delta -, \mathtt{G}^{\mathrm{presh}} X).
\end{aligned}$$

$\square$

Example 8.4. Consider the adjoint pair $(R^{(-)}, \mathtt{f})$ between $\mathrm{Mod}\, R$ and **Set** from Example 6.3. We note that

$$R^{(X \coprod Y)} = R^{(X)} \oplus R^{(Y)} \text{ and } \mathtt{f}(M \oplus N) = \mathtt{f}M \times \mathtt{f}N$$

by Theorem 8.3 above. (Of course in this example we could also have checked that directly.)

However in general neither

$$R^{(X\times Y)} = R^{(X)} \times R^{(Y)} \text{ nor } \mathtt{f}(M \oplus N) = \mathtt{f}M \coprod \mathtt{f}N.$$

Chapter II

Additive and abelian categories

9 Additive categories

Definition 9.1. A *pre-additive category* is a category $\mathscr{A}$ such that all Hom-sets are abelian groups, and composition of morphisms is bilinear.

An *additive category* is a pre-additive category $\mathscr{A}$ such that

- there is a *zero-object*, i.e. an object 0 such that for any $X \in \mathcal{O}\mathfrak{b}\,\mathscr{A}$ both $\mathrm{Hom}_{\mathscr{A}}(X, 0)$ and $\mathrm{Hom}_{\mathscr{A}}(0, X)$ contain precisely one morphism.
- for any $X, Y \in \mathcal{O}\mathfrak{b}\,\mathscr{X}$ there is a *biproduct*, i.e. an object $X \oplus Y$ with morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "\iota_X"] & X \oplus Y \arrow[l, bend left, "\pi_X"] \arrow[r, bend right, "\pi_Y"'] & Y \arrow[l, bend right, "\iota_Y"'] \end{tikzcd}$$

  such that

$$\mathrm{id}_X = \pi_X \circ \iota_X, \quad \mathrm{id}_Y = \pi_Y \circ \iota_Y, \text{ and } \quad \mathrm{id}_{X \oplus Y} = \iota_X \circ \pi_X + \iota_Y \circ \pi_Y.$$

Example 9.2. • **Ab** is an additive category.

- For a ring R, the category $\operatorname{Mod} R$ is additive.
- **Set** and **Top** are not additive categories.
- For any small category $\mathscr{X}$, and any additive category $\mathscr{A}$, the category $\operatorname{presh}_{\mathscr{A}} \mathscr{X}$ is additive.

Observation 9.3. In the situation of the biproduct diagram, we have

$$\pi_Y \circ \iota_X = \pi_Y \circ \underbrace{\iota_X \circ \pi_X}_{=\mathrm{id}_{X\oplus Y} - \iota_Y \circ \pi_Y} \circ \iota_X = \pi_y \circ \iota_X - \underbrace{\pi_Y \circ \iota_Y}_{=\mathrm{id}_Y} \circ \pi_Y \circ \iota_X = 0,$$

and similarly

$$\pi_X \circ \iota_Y = 0.$$

Lemma 9.4. *Let $\mathscr{A}$ be an additive category. Then, for any two objects X and Y, the biproduct $X \oplus Y$ is a product and a coproduct of X and Y.*

Proof. We show that $X \oplus Y$ is a product, the proof that it is a coproduct is dual.

We have to show that for any maps $f_X \colon H \longrightarrow X$ and $f_Y \colon H \longrightarrow Y$ there is precisely one map $f \colon H \longrightarrow X \oplus Y$ such that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$.

We see that

$$f = \mathrm{id}_{X\oplus Y} \circ f = \iota_X \circ \pi_X \circ f + \iota_Y \circ \pi_Y \circ f = \iota_X \circ f_X + \iota_Y \circ f_Y.$$

Thus f is unique. On the other hand we can see that $\iota_X \circ f_X + \iota_Y \circ f_Y$ fullfils the requirements:

$$\pi_X \circ (\iota_X \circ f_X + \iota_Y \circ f_Y) = \underbrace{\pi_X \circ \iota_X}_{=\mathrm{id}_X} \circ f_X + \underbrace{\pi_X \circ \iota_Y}_{=0} \circ f_Y = f_X.$$

and similarly

$$\pi_Y \circ (\iota_X \circ f_X + \iota_Y \circ f_Y) = f_Y.$$

□

Remark 9.5.
- In particular in an additive category any two objects have isomorphic product and coproduct. This shows that neither **Set** nor **Top** can be additive categories.

- It is possible to show that the addition of morphism is completely determined by the biproducts, and not an additional part of the structure.

  That is, an additive category is a category with a zero-object, such that any two objects have a product and a coproduct which are isomorphic, satisfying certain properties.

Remark 9.6. For $n \geqslant 1$, and objects $X_1, \dots, X_n$, we can iteratedly construct

$$X = (\cdots(X_1 \oplus X_2) \oplus X_3) \cdots) \oplus X_n.$$

We note that for this object we have, similarly to the biproduct diagram and with maps given by compositions of the maps there

$$\pi_i \colon X \longrightarrow X_i, \text{ and } \iota_i \colon X_i \longrightarrow X$$

such that

$$\pi_i \circ \iota_i = \mathrm{id}_{X_i} \ \forall i, \text{ and } \sum_{i=1}^{n} \iota_i \circ \pi_i = \mathrm{id}_X .$$

Remark 9.7 (Matrix notation)**.** We often use the following intuitive matrix notation for morphisms from $X = X_1 \oplus \cdots \oplus X_n$ to $Y = Y_1 \oplus \cdots \oplus Y_m$:

A morphism $f \colon X \longrightarrow Y$ is represented by the matrix

$$(\pi_{Y_i} \circ f \circ \iota_{X_j})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}.$$

Conversely, given a matrix

$$(f_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \text{ with } f_{ij} \colon X_j \longrightarrow Y_i$$

we can interpret it as the map

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \iota_{Y_i} \circ f_{ij} \circ \pi_{X_j} \colon X \longrightarrow Y.$$

One easily sees that these constructions are mutually inverse to each other, and thus we may identify matrices and maps between biproducts.

The main advantage of this notation is, that composition of maps is just given by matrix multiplication:

Given

$$\bigoplus_{k=1}^{o} X_k \xrightarrow{(f_{jk})} \bigoplus_{j=1}^{n} Y_j \xrightarrow{(g_{ij})} \bigoplus_{i=1}^{m} Z_i$$

we have

$$\begin{aligned}(g_{ij})_{i,j} \circ (f_{jk})_{j,k} &= (\sum_{i=1}^{m}\sum_{j=1}^{n} \iota_{Z_i} \circ g_{ij} \circ \pi_{Y_j}) \circ (\sum_{j=1}^{n}\sum_{k=1}^{o} \iota_{Y_j} \circ f_{jk} \circ \pi_{X_k}) \\ &= (\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o} \iota_{Z_i} \circ g_{ij} \circ f_{jk} \circ \pi_{X_k}) \\ &= (\sum_{j=1}^{n} g_{ij} \circ f_{jk})_{i,k}.\end{aligned}$$

10 Kernels and cokernels

Definition 10.1. Let $f\colon X \longrightarrow Y$ be a morphism in an additive category. The *kernel* of f is (if it exists) the pullback of

$$\begin{tikzcd} & 0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In other words, the kernel is given by an object $\operatorname{Ker} f$, together with a morphism $\kappa\colon \operatorname{Ker} f \longrightarrow X$ (the other morphism $\operatorname{Ker} f \longrightarrow 0$ necessarily being 0), such that $f \circ \kappa = 0$, and that for any object H and morphism $h\colon H \longrightarrow X$ such that $f \circ h = 0$ there is a unique morphism $\widehat{h}\colon H \longrightarrow \operatorname{Ker} f$ such that $h = \kappa \circ \widehat{h}$.

Dually, the *cokernel* of f is, if it exists, the pushout of

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

and consists of an object $\operatorname{Cok} f$ and a map $\pi\colon Y \longrightarrow \operatorname{Cok} f$.

Observation 10.2. In the definition of kernel above the map κ is a monomorphism: Let $h, g\colon H \longrightarrow \operatorname{Ker} f$ such that $\kappa \circ h = \kappa \circ g$. Then clearly $f \circ \kappa \circ h = 0$, and therefore $\kappa \circ h$ factors *uniquely* through κ, i.e. $h = g$.

Dually the map π in the definition of cokernel is an epimorphism.

Lemma 10.3. *Let $f\colon X \longrightarrow Y$ be a morphism in an additive category. Then f is a monomorphism if and only if $0 \longrightarrow X$ is a kernel of f. Dually f is an epimorphism if and only if $Y \longrightarrow 0$ is a cokernel of f.*

Proof. Assume first that f is a monomorphism. Then any morphism $h\colon H \longrightarrow X$ such that $f \circ h = 0$ is necessarily 0, and therefore factors (uniquely) through $0 \longrightarrow X$.

Conversely, assume $0 \longrightarrow X$ is a kernel of f. Then any map h such that $f \circ h = 0$ factors through 0, that is is zero. □

11 Abelian categories

Definition 11.1. A *pre-abelian category* is an additive category $\mathscr{A}$, in which every morphism has a kernel and a cokernel.

Definition 11.2. Let $\mathscr{A}$ be pre-abelian, and $f\colon X \longrightarrow Y$ a morphism. Let $\operatorname{Ker} f \xrightarrow{\iota} X$ and $Y \xrightarrow{\pi} \operatorname{Cok} f$ be kernel and cokernel of f. Then

- the *image* of f, denoted by $\operatorname{Im} f$, is the kernel of π;
- the *coimage* of f, denoted by $\operatorname{Coim} f$, is the cokernel of ι.

Proposition 11.3. *In the setup of Definition 11.2, there is a unique map $\overline{f}$ making the diagram*

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\rho"'] & & & Y \arrow[r, two heads, "\pi"] & \operatorname{Cok} f \\
& & \operatorname{Coim} f \arrow[r, dashed, "\overline{f}"] & \operatorname{Im} f \arrow[ur, tail, "\kappa"'] & &
\end{tikzcd}$$

commutative.

Proof. Uniqueness of $\overline{f}$ follows immediately, since morphisms ρ and κ are epi and mono, respectively.

Since $f \circ \iota = 0$ there is a morphism $f'\colon \operatorname{Coim} f \longrightarrow Y$ such that $f' \circ \rho = f$. Moreover, since ρ is epi, $0 = \pi \circ f = \pi \circ f' \circ \rho$ implies $\pi \circ f' = 0$, hence f' factors through κ. This proves the existence of $\overline{f}$. □

Definition 11.4. An *abelian category* is a pre-abelian category, in which, for any morphism $f\colon X \longrightarrow Y$ the induced morphism $\overline{f}\colon \operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is an isomorphism.

Remark 11.5. In other words, an abelian category is an additive category with kernels and cokernels, in which the first isomorphism theorem holds. (Recall that the first isomorphism theorem is precisely that X modulo kernel is isomorphic with the image.)

Observation 11.6. In an abelian category

- every monomorphism is a kernel of its cokernel;
- every epimorphism is a cokernel of its kernel;
- every morphism that is both a monomorphism and an epimorphism is an isomorphism.

Remark 11.7. One can show that the first two points above give an equivalent definition of abelian category.

12 Exact sequences, pullbacks and pushouts

Observation 12.1. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in an abelian category, such that $g \circ f = 0$. Then we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Im} f \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d, equal] & \operatorname{Cok} f \arrow[d] \\
\operatorname{Ker} g \arrow[r, tail] & B \arrow[r, two heads] & \operatorname{Im} g
\end{tikzcd}$$

where the right part consists of the cokernels of the left horizontal maps, and the left part consists of the kernels of the right horizontal maps.

It follows that the morphism $\operatorname{Im} f \longrightarrow \operatorname{Ker} g$ is an isomorphism if and only if the morphism $\operatorname{Cok} f \longrightarrow \operatorname{Im} g$ is. (We may note that the former always is a monomorphism, and the latter always is an epimorphism.)

Definition 12.2. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in an abelian category, such that $g \circ f = 0$. We say that this sequence of morphisms is *exact* if the natural morphism $\operatorname{Im} f \longrightarrow \operatorname{Ker} g$ is an isomorphism.

We say that a longer sequence of morphisms is exact if it is exact in every (inner) position.

Example 12.3.

- The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if $\operatorname{Ker} f = 0$, that is if and only if f is a monomorphism.
- Dually the sequence $A \xrightarrow{g} B \longrightarrow 0$ is exact if and only if $\operatorname{Cok} g = 0$.
- The sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if
  - f is a monomorphism (as before), and
  - $\operatorname{Ker} g = \operatorname{Im} f = A$, that is $A \xrightarrow{f} B$ is a kernel of g.
- Dually the sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if $B \xrightarrow{g} C$ is a cokernel of f.
- The sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if both f is a kernel of g and g is a cokernel of f. Such an exact sequence is called *short exact sequence*.

Proposition 12.4. *Let $\mathscr{A}$ be an abelian category. Consider the morphisms in the following commutative square.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "i"] & D
\end{tikzcd}$$

- *The square is a pullback if and only if the sequence*

$$0 \longrightarrow A \xrightarrow{\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)} B \oplus C \xrightarrow{(h \quad i)} D$$

  is exact.

- *The square is a pushout if and only if the sequence*

$$A \xrightarrow{\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)} B \oplus C \xrightarrow{(h \quad i)} D \longrightarrow 0$$

  is exact.

Proof. We first observe that the commutativity of the square means that $(h \quad i) \circ \left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right) = -h \circ f + i \circ g = 0$.

Now observe that the square is a pullback if and only if

$$\begin{aligned}&\forall \widetilde{A} \in \mathcal{O}b\,\mathscr{A}\ \forall \widetilde{f} \colon \widetilde{A} \longrightarrow B\ \forall \widetilde{g} \colon \widetilde{A} \longrightarrow C \colon\\ &\qquad \text{if } h \circ \widetilde{f} = i \circ \widetilde{g} \text{ then } \exists! \varphi \colon \widetilde{A} \longrightarrow A \colon \widetilde{f} = f \circ \varphi \text{ and } \widetilde{g} = g \circ \varphi\end{aligned}$$

assembling maps in matrices we obtain that this is equivalent to

$$\begin{aligned}&\forall \widetilde{A} \in \mathcal{O}b\,\mathscr{A}\ \forall \begin{pmatrix}-\widetilde{f}\\ \widetilde{g}\end{pmatrix} \colon \widetilde{A} \longrightarrow B \oplus C \colon\\ &\qquad \text{if } (h \quad i) \circ \begin{pmatrix}-\widetilde{f}\\ \widetilde{g}\end{pmatrix} = 0 \text{ then } \exists! \varphi \colon \widetilde{A} \longrightarrow A \colon \begin{pmatrix}-\widetilde{f}\\ \widetilde{g}\end{pmatrix} = \left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right) \circ \varphi\end{aligned}$$

Now note that this last statement is precisely the definition of a kernel.

The proof of the second point is dual. □

Remark 12.5. Proposition 12.4 shows, in particular, that in abelian categories pullbacks and pushouts always exist.

Corollary 12.6. *Let $\mathscr{A}$ be an additive category. If the square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

- *is a pullback, and i is an epi, then it is also a pushout;*
- *is a pushout, and f is a mono, then it is also a pullback.*

Proposition 12.7. *Let $\mathscr{A}$ be an abelian category.*

- *If the square*
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$
is a pullback, then the kernel morphism $\operatorname{Ker} f \longrightarrow \operatorname{Ker} i$ is an isomorphism.
- *If the square is a pushout then the cokernel morphism $\operatorname{Cok} f \longrightarrow \operatorname{Cok} i$ is an isomorphism.*

Proof. We only prove the first part, the second one is dual.

Denote the inclusions of the kernels by $\iota\colon \operatorname{Ker} f \rightarrowtail A$ and $\kappa\colon \operatorname{Ker} i \rightarrowtail C$, respectively, and the kernel morphism by $\varphi\colon \operatorname{Ker} f \longrightarrow \operatorname{Ker} i$. Consider the morphism $0\colon \operatorname{Ker} i \longrightarrow B$, as indicated in the following diagram.

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, tail, "\iota"] \arrow[d, "\varphi"'] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ \operatorname{Ker} i \arrow[r, tail, "\kappa"'] \arrow[u, dashed, bend right, "\widehat{\widehat{\kappa}}"'] \arrow[ur, dashed, "\widehat{\kappa}"] \arrow[urr, "0"] & C \arrow[r, "i"] & D \end{tikzcd}$$

Clearly $i \circ \kappa = 0 = h \circ 0$, so by the pullback property there is a morphism $\widehat{\kappa}\colon \operatorname{Ker} i \longrightarrow A$ such that $\kappa = g \circ \widehat{\kappa}$ and $0 = f \circ \widehat{\kappa}$. By the second equality $\widehat{\kappa}$ factors through the kernel of f, that is there is a morphism $\widehat{\widehat{\kappa}}\colon \operatorname{Ker} i \longrightarrow \operatorname{Ker} f$ such that $\widehat{\kappa} = \iota \circ \widehat{\widehat{\kappa}}$.

Now it only remains to verify that $\widehat{\widehat{\kappa}}$ is an inverse of φ. Firstly we have

$$\kappa \circ \varphi \circ \widehat{\widehat{\kappa}} = g \circ \iota \circ \widehat{\kappa} = g \circ \widehat{\kappa} = \kappa,$$

and hence, since κ is a monomorphism,

$$\varphi \circ \widehat{\widehat{\kappa}} = \mathrm{id}_{\operatorname{Ker} i}\,.$$

Secondly we have

$$\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)\circ\iota\circ\widehat{\widehat{\kappa}}\circ\varphi=\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)\circ\widehat{\kappa}\circ\varphi=\left(\begin{smallmatrix}0\\ \kappa\end{smallmatrix}\right)\circ\varphi=\left(\begin{smallmatrix}0\\ \kappa\circ\varphi\end{smallmatrix}\right)=\left(\begin{smallmatrix}-f\circ\iota\\ g\circ\iota\end{smallmatrix}\right)=\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)\circ\iota,$$

and hence, since both ι and $\left(\begin{smallmatrix}-f\\ g\end{smallmatrix}\right)$ are monomorphisms,

$$\widehat{\widehat{\kappa}}\circ\varphi=\mathrm{id}_{\operatorname{Ker} f}\,.$$

□

Corollary 12.8. *In an abelian category*

- *the pullback of a mono is a mono;*
- *the pullback of an epi is an epi;*
- *the pushout of a mono is a mono;*
- *the pushout of an epi is an epi.*

Moreover, in the case of the second and third point, the square in question is actually both a pullback and a pushout.

Proof. The first point follows immediately from Proposition 12.7 above.

For the second point, note first that the pullback now also is a pushout, by Corollary 12.6. Now apply (the dual-part of) Proposition 12.7.

The third and fourth points are dual to the second and first, respectively. □

Proposition 12.9. *In an abelian category, let $A\xrightarrow{f}B\xrightarrow{g}C$ be morphisms such that $g\circ f=0$. Then the following are equivalent.*

- *The sequence $A\xrightarrow{f}B\xrightarrow{g}C$ is exact.*
- *For any morphism $x\colon X\longrightarrow B$, such that $g\circ x=0$, there are an object $\widehat{X}$, and morphisms $\widehat{x}$ and $\widehat{f}$ as in the following diagram*

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{x}"'] & X \arrow[d, "x"'] \arrow[dr, dotted, bend left] & \\
A \arrow[r, "f"] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

such that the square commutes, and $\widehat{f}$ is an epimorphism.

- *For any morphism* $y\colon B \longrightarrow Y$*, such that* $y \circ f = 0$*, there are an object* $\check{Y}$*, and morphisms* $\check{y}$ *and* $\check{f}$ *as in the following diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dotted, bend right] & B \arrow[r, "g"] \arrow[d, "y"] & C \arrow[d, "\check{y}"] \\
& Y \arrow[r, "\check{g}"] & \check{Y}
\end{tikzcd}
$$

such that the square commutes, and $\check{g}$ *is a monomorphism.*

Proof. We only prove the equivalence of the first two points. The equivalence of the first and last point is dual to this.

Assume first that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact, that is $\mathrm{Im}\, f \rightarrowtail B$ is a kernel of g. Thus any morphism x such that $g \circ x = 0$ factors through $\mathrm{Im}\, f \rightarrowtail B$, as indicated in the following diagram.

$$
\begin{tikzcd}
& X \arrow[d, "x'"] & \\
A \prod\limits_{\mathrm{Im}\, f} X \arrow[ur, "\widehat{f'}"] \arrow[d, "\widehat{x'}"'] & \mathrm{Im}\, f \arrow[d, tail] \arrow[dr, dotted, bend left] & \\
A \arrow[ur, "f'"] \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

We form the pullback as indicated above. By Corollary 12.8 the morphism $\widehat{f'}$ is epi.

Now assume conversely that the second point holds. In particular we can find a commutative diagram

$$
\begin{tikzcd}
\widehat{\mathrm{Ker}\, g} \arrow[r, two heads, "\widehat{f}"] \arrow[d, "\widehat{\iota}"'] & \mathrm{Ker}\, g \arrow[d, "\iota"'] \arrow[dr, dotted, bend left] & \\
A \arrow[r, "f"] & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

where $\iota\colon \operatorname{Ker} g \longrightarrow B$ is a kernel of g, and $\widehat{f}$ is an epimorphism.

Then $\operatorname{Ker} g$ is the image of $\iota \circ \widehat{f} = f \circ \widehat{\iota}$, and the inclusion of $\operatorname{Ker} g$ into B factors through the inclusion of $\operatorname{Im} f$. It follows that the inclusion of $\operatorname{Im} f$ into $\operatorname{Ker} g$, which exists since $g \circ f = 0$, is an isomorphism, i.e. that the sequence is exact. □

Remark 12.10. In the category $\operatorname{Mod} R$ of modules over a ring we can determine exactness using elements: A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $g \circ f = 0$ is exact if for every element $x \in B$ such that $g(x) = 0$ there is a preimage, that is $\widehat{x} \in A$ such that $f(\widehat{x}) = x$.

Proposition 12.9 now tells us that the same holds for arbitrary abelian categories, if we replace "element" by "morphism $X \xrightarrow{x}$", and "preimage" by "commutative square with epimorphism".

We will see this kind of substitution in practice in the next section.

13 Some diagram lemmas

Theorem 13.1 (Five lemma). *Let $\mathscr{A}$ be an abelian category. Consider the following commutative diagram with exact rows.*

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5
\end{tikzcd}$$

- *Assume f_2 and f_4 are monomorphisms, and f_1 is an epimorphism. Then f_3 is a monomorphism.*
- *Assume f_2 and f_4 are epimorphisms, and f_5 is a monomorphism. Then f_3 is an epimorphism.*

In particular, if all of f_1, f_2, f_4, and f_5 are isomorphisms, then so is f_3.

Proof for the case Mod R.
FIRST POINT: Let $x \in A_3$, such that $f_3(x) = 0$. Then $f_4(a_3(x)) = b_3(f_3(x)) = 0$, and, since f_4 is a monomorphism, $a_3(x) = 0$.

Thus there is a preimage $\widehat{x}$ of x in A_2. We see that $b_2(f_2(\widehat{x})) = f_3(a_2(\widehat{x})) = f_3(x) = 0$, and thus there is a preimage $\widehat{f_2(\widehat{x})}$ of $f_2(\widehat{x})$ in B_1.

Since f_1 is assumed to be an epimorphism we can find a preimage $\widetilde{f_2(\widehat{x})}$ of $\widehat{f_2(\widehat{x})}$ in A_1.

Now note that

$$f_2(a_1(\widetilde{f_2(\widehat{x})})) = b_1(f_1(\widetilde{f_2(\widehat{x})})) = f_2(\widehat{x}),$$

and, since f_2 is a monomorphism this implies $a_1(\widetilde{f_2(\widehat{x})}) = \widehat{x}$.

Thus $x = a_2(\widehat{x}) = a_2(a_1(\widetilde{f_2(\widehat{x})})) = 0$.
SECOND POINT: Let $x \in B_3$. Since f_4 is epi there is $x' \in A_4$ such that $f_4(x') = b_3(x)$.

We note that $f_5(a_4(x')) = b_4(f_4(x')) = b_4(b_3(x)) = 0$. Thus, since f_5 is mono, we have $a_4(x') = 0$. It follows that there is $\widehat{x} \in A_3$ such that $a_3(\widehat{x}) = x'$.

Next observe that

$$b_3(x - f_3(\widehat{x})) = b_3(x) - b_3(f_3(\widehat{x})) = b_3(x) - f_4(\underbrace{a_3(\widehat{x})}_{=x'}) = 0.$$

Hence there is $y \in B_2$ such that $b_2(y) = x - f_3(\widehat{x})$. Moreover, since f_2 is epi, there is $\widehat{y} \in A_2$ such that $f_2(\widehat{y}) = y$.

Now we have that

$$f_3(\widehat{x} + a_2(\widehat{y})) = f_3(\widehat{x}) + b_2(\underbrace{f_2(\widehat{y})}_{=y}) = f_3(\widehat{x}) + x - f_3(\widehat{x}) = x,$$

showing that an arbitrary x lies in the image of f_3. $\square$

Proof for arbitrary abelian categories. We only prove the first point, the second one is dual.

Let $x \colon X \longrightarrow A_3$ be a kernel of f_3. Since

$$f_4 \circ a_3 \circ x = b_3 \circ f_3 \circ x = 0,$$

and f_4 is mono by assumption, we have $a_3 \circ x = 0$. Thus, by Proposition 12.9, we obtain $\widehat{X}$, $\widehat{x}$, and an epimorphism $\widehat{a_2}$ as indicated in the following diagram.

$$\begin{tikzcd}
\widehat{\widehat{X}} \arrow[r, two heads, "\widehat{b_1 \circ f_1}"] \arrow[d, "\widehat{f_2 \circ \widehat{x}}"'] & \widehat{X} \arrow[r, two heads, "\widehat{a_2}"] \arrow[d, "\widehat{x}"'] & X \arrow[d, "x"'] \\
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[d, "f_3"'] \\
B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3
\end{tikzcd}$$

Now note that $b_2 \circ (f_2 \circ \widehat{x}) = 0$, and that, since f_1 is epi,

$$A_1 \xrightarrow{b_1 \circ f_1} B_2 \xrightarrow{b_2} B_3$$

is exact. Hence we can find $\widehat{\widehat{X}}$, $\widehat{f_2 \circ \widehat{x}}$, and an epimorphism $\widehat{b_1 \circ f_1}$ as indicated above, such that $b_1 \circ f_1 \circ \widehat{f_2 \circ \widehat{x}} = f_2 \circ \widehat{x} \circ \widehat{b_1 \circ f_1}$. Since $b_1 \circ f_1 = f_2 \circ a_1$, and f_2 is a monomorphism by assumption, this implies

$$a_1 \circ \widehat{f_2 \circ \widehat{x}} = \widehat{x} \circ \widehat{b_1 \circ f_1},$$

and thus

$$x \circ \widehat{a_2} \circ \widehat{b_1 \circ f_1} = \underbrace{a_2 \circ a_1}_{=0} \circ \widehat{f_2 \circ \widehat{x}} = 0.$$

Since $\widehat{a_2} \circ \widehat{b_1 \circ f_1}$ is an epimorphism this means that $x = 0$.

Thus we have seen that $f_3 \circ x = 0$ impies $x = 0$, which means that f_3 is a monomorphism. $\square$

Theorem 13.2 (Characterization of pullback and pushout)**.** *In an abelian category, consider a commutative square, together with its kernel and cokernel morphisms as in the following diagram.*

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "\iota"] \arrow[d, "k"'] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, two heads, "\pi"] \arrow[d, "h"'] & \operatorname{Cok} f \arrow[d, "c"'] \\
\operatorname{Ker} i \arrow[r, tail, "\kappa"] & C \arrow[r, "i"] & D \arrow[r, two heads, "\rho"] & \operatorname{Cok} i
\end{tikzcd}$$

Then

- *the square is a pullback if and only if k is an isomorphism and c is a monomorphism;*

- *the square is a pushout if and only if k is an epimorphism and c is an isomorphism.*

Proof. We only prove the first claim, the second one is dual.

Assume first that the square is a pullback. We have already seen – in Proposition 12.7 – that the kernel morphism k is an isomorphism. Let $x\colon X \longrightarrow \operatorname{Cok} f$ be a morphism such that $c \circ x = 0$.

$$\begin{tikzcd}
\widehat{\widehat{X}} \arrow[rr, two heads, "\widehat{i}"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, "\widehat{h\circ\widehat{x}}"'] & & \widehat{X} \arrow[r, two heads, "\widehat{\pi}"] \arrow[d, "\widehat{x}"'] & \widehat{X} \arrow[d, tail, "x"'] \\
& A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[r, two heads, "\pi"] \arrow[d, "h"'] & \operatorname{Cok} f \arrow[d, "c"'] \\
& C \arrow[r, "i"] & D \arrow[r, two heads, "\rho"] & \operatorname{Cok} i
\end{tikzcd}$$

Since $B \xrightarrow{\pi} \operatorname{Cok} f \longrightarrow 0$ is exact, by Proposition 12.9, there are $\widehat{X}$, $\widehat{x}$, and an epimorphism $\widehat{\pi}$ as indicated in the diagram.

Since $C \xrightarrow{i} D \xrightarrow{\rho} \operatorname{Cok} i$ is exact, and $\rho\circ(h\circ\widehat{x}) = c\circ x\circ\widehat{\pi} = 0$, Proposition 12.9 also implies the existence of $\widehat{\widehat{X}}$, $\widehat{h\circ\widehat{x}}$, and an epimorphism $\widehat{i}$ as above.

We get φ as indicated above by the pullback property of the original square.

Thus $x \circ \widehat{\pi} \circ \widehat{i} = \pi \circ f \circ \varphi = 0$ implies $x = 0$, and thus c is a monomorphism.

Now assume conversely that k is an isomorphism, and c a monomorphism. We have to show that the square is a pullback.

We consider the pullback of i and h, and, by the pullback property, we get a map φ to it from A as indicated in the following diagram.

$$
\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "\iota"] \arrow[d, "k''"] \arrow[dd, bend right, "k"', "\cong"] & A \arrow[r, "f"] \arrow[d, "\varphi"] \arrow[dd, bend right, "g"'] & B \arrow[r, two heads, "\pi"] \arrow[d, "\mathrm{id}_B"'] & \operatorname{Cok} f \arrow[d, "c''"'] \arrow[dd, tail, bend left, "c"] \\
\operatorname{Ker} \widehat{i} \arrow[r, tail, "z"] \arrow[d, "k'"] & B \prod\limits_D C \arrow[r, "\widehat{i}"] \arrow[d, "\widehat{h}"] & B \arrow[r, two heads, "x"] \arrow[d, "h"'] & \operatorname{Cok} \widehat{i} \arrow[d, "c'"'] \\
\operatorname{Ker} i \arrow[r, tail, "\kappa"] & C \arrow[r, "i"] & D \arrow[r, two heads, "\rho"] & \operatorname{Cok} i
\end{tikzcd}
$$

By the other implication of this theorem, we know that k' is an isomorphism. It follows that also k'' is an isomorphism. Moreover, since c is mono, so is c''.

It now follows from the five lemma (Theorem 13.1) that φ is an isomorphism. □

Theorem 13.3 (Snake lemma). *In an abelian category, consider (solid part of) the following diagram with exact rows and columns*

$$
\begin{tikzcd}
 & \operatorname{Ker} f_1 \arrow[r, dashed] \arrow[d, tail, "\iota_1"'] & \operatorname{Ker} f_2 \arrow[r, dashed] \arrow[d, tail, "\iota_2"'] & \operatorname{Ker} f_3 \arrow[d, tail, "\iota_3"'] \arrow[dddll, dashed, rounded corners, "\partial"] & \\
 & A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "b_1"'] \arrow[d, two heads, "\pi_1"] & B_2 \arrow[r, "b_2"'] \arrow[d, two heads, "\pi_2"] & B_3 \arrow[d, two heads, "\pi_3"] & \\
 & \operatorname{Cok} f_1 \arrow[r, dashed] & \operatorname{Cok} f_2 \arrow[r, dashed] & \operatorname{Cok} f_3 &
\end{tikzcd}
$$

Then there is a map $\partial\colon \operatorname{Ker} f_3 \longrightarrow \operatorname{Cok} f_1$*, such that the dashed sequence*

$$
\operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2 \longrightarrow \operatorname{Ker} f_3 \xrightarrow{\partial} \operatorname{Cok} f_1 \longrightarrow \operatorname{Cok} f_2 \longrightarrow \operatorname{Cok} f_3
$$

is exact.

Proof. Construction of ∂: Consider the pullback $A_2 \prod\limits_{A_3} \operatorname{Ker} f_3$, and the pushout $B_2 \coprod\limits_{B_1} \operatorname{Cok} f_1$. By Theorem 13.2 we have induced exact sequences

$$A_1 \xrightarrow{\widehat{a_1}} A_2 \prod_{A_3} \operatorname{Ker} f_3 \xrightarrow{\widehat{a_2}} \operatorname{Ker} f_3 \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Cok} f_1 \xrightarrow{\widetilde{b_1}} B_2 \coprod_{B_1} \operatorname{Cok} f_1 \xrightarrow{\widetilde{b_2}} B_3,$$

as indicated in the following diagram.

$$\begin{tikzcd}
\operatorname{Ker} f_1 \arrow[rr, "k_1"] \arrow[dd, tail, "\iota_1"'] & & \operatorname{Ker} f_2 \arrow[rr, "k_2"] \arrow[dd, tail, "\iota_2"'] \arrow[dr] & & \operatorname{Ker} f_3 \arrow[dd, tail, "\iota_3"'] \arrow[lllldddddd, dashed, bend left=60, "\partial"] \\
& & & A_2 \prod\limits_{A_3} \operatorname{Ker} f_3 \arrow[ur, "\widehat{a_2}"'] \arrow[dl, "\widehat{\iota_3}"] & \\
A_1 \arrow[rr, "a_1"'] \arrow[urrr, "\widehat{a_1}"] \arrow[d, "f_1"'] & & A_2 \arrow[rr, two heads, "a_2"'] \arrow[d, "f_2"'] & & A_3 \arrow[d, "f_3"'] \\
B_1 \arrow[rr, tail, "b_1"] \arrow[dd, two heads, "\pi_1"] & & B_2 \arrow[rr, "b_2"] \arrow[dl, "\widetilde{\pi_1}"'] \arrow[dd, two heads, "\pi_2"] & & B_3 \arrow[dd, two heads, "\pi_3"] \\
& B_2 \coprod\limits_{B_1} \operatorname{Cok} f_1 \arrow[uurrr, "\widetilde{b_2}"'] \arrow[dr] & & & \\
\operatorname{Cok} f_1 \arrow[rr, "c_1"'] \arrow[ur, tail, "\widetilde{b_1}"] & & \operatorname{Cok} f_2 \arrow[rr, "c_2"'] & & \operatorname{Cok} f_3
\end{tikzcd}$$

We consider the composition

$$\widetilde{\pi_1} \circ f_2 \circ \widehat{\iota_3}$$

in the middle of the diagram.

Note that both

$$(\widetilde{\pi_1} \circ f_2 \circ \widehat{\iota_3}) \circ \widehat{a_1} = \widetilde{b_1} \circ \pi_1 \circ f_1 = 0$$

and

$$\widetilde{b_2} \circ (\widetilde{\pi_1} \circ f_2 \circ \widehat{\iota_3}) = f_3 \circ \iota_3 \circ \widehat{a_2} = 0.$$

Thus $\widetilde{\pi_1} \circ f_2 \circ \widehat{\iota_3}$ factors through both $\widehat{a_2}$ and $\widetilde{b_1}$, that is we can (uniquely) find ∂ such that

$$\widetilde{b_1} \circ \partial \circ \widehat{a_2} = \widetilde{\pi_1} \circ f_2 \circ \widehat{\iota_3}.$$

Exactness in $\operatorname{Ker} f_2$*:* We first note that $k_2 \circ k_1 = 0$ by functoriality of kernels. Now we aim to apply Proposition 12.9 to show exactness.

Let $x\colon X \longrightarrow \operatorname{Ker} f_2$ such that $k_2 \circ x = 0$. Then clearly also $a_2 \circ \iota_2 \circ x = 0$. Hence, since the sequence $A_1 \longrightarrow A_2 \longrightarrow A_3$ is exact, by Proposition 12.9, there is an object $\widehat{X}$ and morphisms $\widehat{a_1}$ and $\widehat{\iota_2 \circ x}$ such that $\widehat{a_1}$ is epi, and $\widehat{\iota_2 \circ x} \circ a_1 = \widehat{a_1} \circ \iota_2 \circ x$, as indicated in the following diagram.

$$\begin{tikzcd}
\widehat{X} \arrow[rr, two heads, "\widehat{a_1}"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, "\widehat{\iota_2 \circ x}"'] & & X \arrow[d, "x"] & \\
& \operatorname{Ker} f_1 \arrow[r, "k_1"] \arrow[d, tail, "\iota_1"] & \operatorname{Ker} f_2 \arrow[r, "k_2"] \arrow[d, tail, "\iota_2"] & \operatorname{Ker} f_3 \arrow[d, tail, "\iota_3"] \\
& A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \\
0 \arrow[r] & B_1 \arrow[r, "b_1"'] & B_2 &
\end{tikzcd}$$

We note that $b_1 \circ f_1 \circ \widehat{\iota_2 \circ x} = f_2 \circ \iota_2 \circ x \circ \widehat{a_1} = 0$. Since b_1 is mono this implies that $f_1 \circ \widehat{\iota_2 \circ x} = 0$, so $\widehat{\iota_2 \circ x} = \iota_1 \circ \varphi$ for some φ as indicated by the dashed arrow above. Since

$$\iota_2 \circ k_1 \circ \varphi = a_1 \circ \iota_1 \circ \varphi = a_i \circ \widehat{\iota_2 \circ x} = \iota_2 \circ x \circ \widehat{a_1},$$

and since ι_2 is a monomorphism, it follows that also the upper square of the diagram commutes. Now exactness in $\operatorname{Ker} f_2$ follows from Proposition 12.9.

Exactness in $\operatorname{Ker} f_3$*:* We begin by noting that $\partial \circ k_2 = 0$: This follows form $\widetilde{b_1} \circ \partial \circ k_2 = \widetilde{\pi_1} \circ f_2 \circ \iota_2 = 0$, since $\widetilde{b_1}$ is a monomorphism.

Now we proceed showing exactness by using Proposition 12.9. Thus, let $y\colon \operatorname{Ker} f_3 \longrightarrow Y$ such that $y \circ k_2 = 0$. We construct the following diagram form top to bottom:

$$\begin{tikzcd}
 & \operatorname{Ker} f_2 \arrow[r, "k_2"] \arrow[d, tail, "\iota_2"'] & \operatorname{Ker} f_3 \arrow[r, "y"] \arrow[d, tail, "\iota_3"'] & Y \arrow[d, tail, "\widetilde{\iota_3}"'] \\
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\widetilde{y}"] & \widetilde{Y} \arrow[d, tail, "\widetilde{f_2}"'] \\
B_1 \arrow[r, "b_1"] \arrow[d, two heads, "\pi_1"'] & B_2 \arrow[rr, "\widetilde{\widetilde{y}}"] & & \widetilde{\widetilde{Y}} \arrow[d, tail, "\widetilde{\pi_1}"'] \\
\operatorname{Cok} f_1 \arrow[rrr, "\widetilde{\widetilde{\widetilde{y}}}"] & & & \widetilde{\widetilde{\widetilde{Y}}}
\end{tikzcd}$$

Here we used Proposition 12.9 thrice:

- $\widetilde{Y}$, $\widetilde{y}$, and a monomorphism $\widetilde{\iota_3}$ exist since ι_3 is a mono;
- $\widetilde{\widetilde{Y}}$, $\widetilde{\widetilde{y}}$, and a monomorphism $\widetilde{f_2}$ exist because $\operatorname{Ker} f_2 \xrightarrow{\iota_2} A_2 \xrightarrow{f_2} B_2$ is exact, and $(\widetilde{y} \circ a_2) \circ \iota_2 = \widetilde{\iota_3} \circ y \circ k_2 = 0$;
- $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{\widetilde{\widetilde{y}}}$, and a monomorphism $\widetilde{\pi_1}$ exist because $A_1 \xrightarrow{f_1} B_1 \xrightarrow{\pi_1} \operatorname{Cok} f_1$ is exact, and $(\widetilde{\widetilde{y}} \circ a_1) \circ f_1 = \widetilde{f_2} \circ \widetilde{y} \circ a_2 \circ a_1 = 0$.

We now claim that

$$(\widetilde{\pi_1} \circ \widetilde{f_2} \circ \widetilde{\iota_3}) \circ y = \widetilde{\widetilde{\widetilde{y}}} \circ \partial.$$

To check this, consider the epimorphism $\widehat{a_2}\colon A_2 \prod_{A_3} \operatorname{Ker} f_3 \longrightarrow \operatorname{Ker} f_3$ as above. Note that we have

$$\partial \circ \widehat{a_2} = \pi_1 \circ h,$$

where h is the unique map $A_2 \prod_{A_3} \operatorname{Ker} f_3 \longrightarrow B_1$ such that $b_1 \circ h = f_2 \circ \widehat{\iota_3}$.

Now we can calculate

$$\begin{aligned}
\widetilde{\widetilde{\widetilde{y}}} \circ \partial \circ \widehat{a_2} &= \widetilde{\widetilde{\widetilde{y}}} \circ \pi_1 \circ h \\
&= \widetilde{\pi_1} \circ \widetilde{\widetilde{y}} \circ b_1 \circ h \\
&= \widetilde{\pi_1} \circ \widetilde{\widetilde{y}} \circ f_2 \circ \widehat{\iota_3} \\
&= \widetilde{\pi_1} \circ \widetilde{f_2} \circ \widetilde{y} \circ a_2 \circ \widehat{\iota_3} \\
&= \widetilde{\pi_1} \circ \widetilde{f_2} \circ \widetilde{y} \circ \iota_3 \circ \widehat{a_2} \\
&= \widetilde{\pi_1} \circ \widetilde{f_2} \circ \widetilde{\iota_3} \circ y \circ \widehat{a_2}.
\end{aligned}$$

And the claim follows since $\widehat{a_2}$ is an epimorphism.

Now exactness of the snake sequence in $\operatorname{Ker} f_3$ follows from Proposition 12.9.

Exactness in $\operatorname{Cok} f_1$ *and* $\operatorname{Cok} f_2$*:* Are dual to the two positions we have already treated. □

Remark 13.4 (Construction of ∂ for $\operatorname{Mod} R$)**.** For an element x of $\operatorname{Ker} f_3$, let $\widehat{x}$ be a preimage of $\iota_3(x)$ in A_2. Then $b_2(f_2(\widehat{x})) = f_3(a_2(\widehat{x})) = f_3(\iota_3(x)) = 0$. Therefore $f_2(\widehat{x})$ has a preimage $\widehat{f_2(\widehat{x})}$ in B_1. We define $\partial(x) = \pi_1(\widehat{f_2(\widehat{x})})$.

One may check that this is well-defined.

Chapter III

Hom **and** $\bigotimes$

14 Hom, projectives and injectives

Let $\mathscr{A}$ be a preadditive category, and $A \in \mathcal{O}\mathfrak{b}\,\mathscr{A}$. Then $\mathrm{Hom}_{\mathscr{A}}(A, -)$ and $\mathrm{Hom}_{\mathscr{A}}(-, A)$ define (additive) functors form $\mathscr{A}$ to **Ab** (covariant and contravariant, respectively).

Now let $\mathscr{A}$ be abelian. We want to investigate what the Hom-functors do to short exact sequences.

Example 14.1. In **Ab**, consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/(2), -)$ we obtain

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/(2) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbf{Ab}}(-, \mathbb{Z}/2)$ we obtain

$$0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{\mathrm{id}} \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \longrightarrow 0.$$

In both cases we observe that the resulting sequence is not exact any more.

However, in both cases we may note that the left map is still the kernel of the right map. We will now see that this is a general feature of Hom-functors.

Theorem 14.2 (Hom is left exact)**.** *Let $\mathscr{A}$ be an abelian category, and let $A \in \mathcal{O}b\,\mathscr{A}$.*

- *Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z$ be exact. Then also*

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, X) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, Y) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, Z)$$

  is exact, that is $\mathrm{Hom}_{\mathscr{A}}(A, -)$ preserves kernels.

- *Let $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be exact. Then also*

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Z, A) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Y, A) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X, A)$$

  is exact, that is $\mathrm{Hom}_{\mathscr{A}}(-, A)$ turns cokernels into kernels.

Such functors are called left exact*.*

Proof. We only prove the first claim, the second one is the same for the category $\mathscr{A}^{\mathrm{op}}$.

We denote by $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ the maps of the original sequence. We first check that $f^* = \mathrm{Hom}_{\mathscr{A}}(A, f)$ is injective. Let $\varphi \in \mathrm{Hom}_{\mathscr{A}}(A, X)$ such that $f^*(\varphi) = 0$. By definition $f^*(\varphi) = f \circ \varphi$, and since f is a monomorphism this can only be zero if φ already is zero.

Next let $\varphi \in \mathrm{Ker}\, g^*$, that is $g \circ \varphi = 0$. Then, since (X, f) is the kernel of g, there is a map $\psi\colon A \longrightarrow X$ such that $\varphi = f \circ \psi$, i.e. $\varphi = f^*(\psi) \in \mathrm{Im}\, f^*$. $\square$

Definition 14.3. A functor between two abelian categories is called *exact* if it preserves short exact sequences. It is called *right exact* if it preserves cokernels.

Observation 14.4. For a functor $\mathtt{F}$ between two abelian categories the following are equivalent:

- $\mathtt{F}$ is exact;
- $\mathtt{F}$ is left exact and maps epimorphisms to epimorphisms;
- $\mathtt{F}$ is right exact and maps monomorphisms to monomorphisms.

Definition 14.5. Let $\mathscr{A}$ be an abelian category.

- An object P is called *projective* if the functor $\mathrm{Hom}_{\mathscr{A}}(P, -)$ is exact.
- An object I is called *injective* if the functor $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is exact.

Clearly injective objects in $\mathscr{A}$ are just projective objects in $\mathscr{A}^{\mathrm{op}}$.

Example 14.6. In the category $\operatorname{Mod} R$, the object R is projective: Indeed the functor $\operatorname{Hom}_R(R, -)\colon \operatorname{Mod} R \longrightarrow \mathbf{Ab}$ is just the forgetful functor, and hence clearly is exact.

Observation 14.7. Direct sums and direct summands of projective objects are projective (and similar for injective). The zero object is projective and injective.

Observation 14.8. Let $\mathscr{A}$ be an abelian category.

- An object P is projective if and only if any given diagram as the solid part of the following, with exact row

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & 0 \end{tikzcd}$$

  can be completed to a commutative diagram by a morphism as indicated by the dashed arrow.

  (This is just a diagrammatic restatement of the fact that the functor $\operatorname{Hom}_{\mathscr{A}}(P, -)$ maps epimorphisms to epimorphisms.

- An object I is injective if and only if any given diagram as the solid part of the following, with exact row

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

  can be completed to a commutative diagram by a morphism as indicated by the dashed arrow.

Recall that the free R-module on a set I is

$$R^{(I)} = \{f\colon I \longrightarrow R \mid f(i) \neq 0 \text{ for only finitely many } i \in I\}.$$

Theorem 14.9. *Let R be a ring, P an R-module. Then the following are equivalent:*

- *P is projective in* $\operatorname{Mod} R$*;*
- *There is a module Q such that $P \oplus Q \cong R^{(I)}$ for some set I.*

Proof. $\Longrightarrow$: Consider the natural map $\pi\colon R^{(P)} \longrightarrow P$ (the counit of the adjunction). It clearly is an epimorphism, and, since P is projective, it splits. Therefore $P \oplus \operatorname{Ker} \pi \cong R^{(P)}$.

$\Longleftarrow$: Since $\operatorname{Hom}_{\operatorname{Mod} R}(R^{(I)}, -) = \operatorname{Hom}_{\mathbf{Set}}(I, -)$, this functor maps epimorphisms to epimorphisms. Hence $R^{(I)}$ is projective. It follows that also all direct summands of $R^{(I)}$ are projective. $\square$

Remark 14.10. It follows that for any R-module M, there is an epimorphism $P \twoheadrightarrow M$ from a projective module. (Take for instance $P = R^{(M)}$.)

It is also possible to show (but a lot more technical) that for every R-module M there is a monomorphism $M \rightarrowtail I$ into an injective R-module.

15 Tensor products

Definition 15.1. Ler R be a ring, M a right R-module and N a left R-module. A map $\varphi\colon M \times N \longrightarrow A$ to an abelian group A is called *R-balanced* if

$$\begin{aligned}
&\forall m \in M, n_1, n_2 \in N\colon \varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2),\\
&\forall m_1, m_2 \in M, n \in N\colon \varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n),\\
&\forall m \in M, n \in N, r \in R\colon \varphi(mr, n) = \varphi(m, rn)
\end{aligned}$$

A *tensor product* is an abelian group $M \otimes_R N$, together with an R-balanced map $t\colon M \times N \longrightarrow M \otimes_R N$, such that for any R-balanced $\varphi\colon M \times N \longrightarrow A$ there is a unique morphism of abelian groups $h\colon M \otimes_R N \longrightarrow A$ such that $\varphi = h \circ t$.

In this situation we write $m \otimes n = t(m, n)$, and call it an *elementary tensor*. Note that there is no reason for t to be surjective in general, that is not all elements of the tensor product need to be elementary tensors.

Theorem 15.2. *Tensor products exist and are unique up to isomorphism.*

Proof. Uniqueness can be shown similarly to the proof of uniqueness of limits and colimits (see Proposition 7.3).

To prove existence we explicitly construct a tensor product. We start by considering the free abelian group $F = \mathbb{Z}^{(M\times N)}$. We have seen in Example 6.3 that

$$\mathrm{Hom}_{\mathbf{Ab}}(F, A) = \mathrm{Hom}_{\mathbf{Set}}(M \times N, A).$$

Now the idea of the proof is that we alter F in such a way that in the right hand side only the R-balanced maps remain. We denote by U the abelian subgroup of F generated by all expressions of the form

$$\chi_{(m,n_1+n_2)} - \chi_{(m,n_1)} - \chi_{(m,n_2)},$$
$$\chi_{(m_1+m_2,n)} - \chi_{(m_1,n)} - \chi_{(m_2,n)}, \text{ and}$$
$$\chi_{(mr,n)} - \chi_{(m,rn)}.$$

Then it is immediately verified that F/U is a tensor product. $\square$

Observation 15.3. The above construction shows that, while not all elements of the tensor product are elementary tensors themselves, they are finite sums of elementary tensors.

Example 15.4. Note that both individual elementary tensors and entire tensor products can be zero, even if they don't "look like it":

Consider $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(3)$. Take an elementary tensor $(a + (2)) \otimes (b + (3))$. Then

$$\begin{aligned}(a + (2)) \otimes (b + (3)) &= (a + (2)) \otimes 2(2b + (3))\\ &= (a + (2))2 \otimes (2b + (3))\\ &= 0 \otimes (2b + (3))\\ &= 0 \otimes 0(2b + (3))\\ &= 0 \otimes 0\end{aligned}$$

Thus all elementary tensors vanish, and hence the entire tensor product is zero.

Construction 15.5. Let $f\colon M_1 \longrightarrow M_2$ be a morphism of right R-modules, and N be a left R-module. Then the composition along the top and right of the following diagram is R-balanced.

$$\begin{tikzcd} M_1 \times N \arrow[r, "f \times \mathrm{id}_N"] \arrow[d] & M_2 \times N \arrow[d] \\ M_1 \otimes_R N \arrow[r, dashed] & M_2 \otimes_R N \end{tikzcd}$$

Thus there is a unique map as indicated by the dashed arrow above, making the diagram commutative. We denote this map by $f \otimes_R N$. One immediately verifies that

$$- \otimes_R N \colon \operatorname{Mod} R \longrightarrow \mathbf{Ab}$$

defines a functor.

Similarly, for a right R-module M one obtains a functor

$$M \otimes_R - \colon \operatorname{Mod} R^{\mathrm{op}} \longrightarrow \mathbf{Ab}.$$

Example 15.6. Let R be any ring, and $M \in \operatorname{Mod} R$. Then

$$M \otimes_R R \cong M.$$

Indeed the map $M \times R \longrightarrow M \colon (m, r) \longmapsto mr$ is clearly R-balanced, thus induces a homomorphism $M \otimes_R R \longrightarrow M$. An inverse is given by $M \longrightarrow M \otimes_R R \colon m \longmapsto m \otimes 1$.

16 Hom-tensor adjunction

Let M be an R-S-bimodule. Then for any R-module L, the tensor product $L \otimes_R M$ becomes an S-module via $(l \otimes m)s = l \otimes ms$. In fact we obtain a functor

$$- \otimes_R M \colon \operatorname{Mod} R \longrightarrow \operatorname{Mod} S.$$

Similarly we have the functor

$$\operatorname{Hom}_S(M, -) \colon \operatorname{Mod} S \longrightarrow \operatorname{Mod} R,$$

where, for an S-module N, the R-module structure on $\operatorname{Hom}_S(M, N)$ is given by $\varphi \cdot r = \varphi(r \cdot -)$.

The following result shows that these two functors are in fact adjoint.

Theorem 16.1. *Let L be an R-module, M be an R-S-bimodule, and N be an S-module. Then*

$$\operatorname{Hom}_S(L \otimes_R M, N) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_S(M, N)),$$

and this isomorphism is natural in all arguments.

Proof. We have mutually inverse maps given by

$$\begin{array}{ccc} \mathrm{Hom}_S(L \otimes_R M, N) & \cong & \mathrm{Hom}_R(L, \mathrm{Hom}_S(M, N)) \\ \varphi & \longmapsto & [\ell \longmapsto \varphi(\ell \otimes m)] \\ [\ell \otimes m \longmapsto \psi(\ell)(m)] & \longleftarrow & \psi \end{array}$$

□

Corollary 16.2. *Let M be an R-S-bimodule. Then the functor*

$$- \otimes_R M \colon \operatorname{Mod} R \longrightarrow \operatorname{Mod} S$$

is right exact.

Proof. Since the functor is right adjoint to $\mathrm{Hom}_S(M, -)$ it commutes with all colimits. But cokernels are certain colimits. □

Remark 16.3. The above argument also shows that tensor products commute with (infinite) coproducts.

Definition 16.4. A left R-module M is called *flat* if the tensor functor $- \otimes_R M \colon \operatorname{Mod} R \longrightarrow \mathbf{Ab}$ is exact.

Observation 16.5.

- The R-module R is flat, since, by Example 15.6 tensoring with R is essentially identity.
- Any free R-module is flat, since, by Remark 16.3 tensoring commutes with coproducts (which are special colimits), and since the coproduct of a collection of exact sequences is exact again.
- Any projective R-module is flat, since it is a direct summand of a free R-module by Theorem 14.9.

Remark 16.6. The converse of the last point above does not hold. For instance $\mathbb{Q}$ is a flat $\mathbb{Z}$ module which is not projective.

However, for certain nice rings (for instance finite dimensional algebras over a field), all flat modules are projective.

Chapter IV

Complexes and homology

17 The long exact sequence of homology

Definition 17.1. Let $\mathscr{A}$ be an abelian category. A (cochain) *complex* in $\mathscr{A}$ is a sequence of objects and morphisms

$$A^\bullet = \cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$.

We denote by $\mathbf{C}(\mathscr{A})$ the category of all complexes in $\mathscr{A}$, where morphisms are given by

$$\begin{aligned}\operatorname{Hom}_{\mathbf{C}(\mathscr{A})}(A^\bullet, B^\bullet) = \{(f^i)_{i\in\mathbb{Z}} \mid f^i \in \operatorname{Hom}_{\mathscr{A}}(A^i, B^i) \text{ such that}\\ f^i \circ d_A^{i-1} = d_B^{i-1} \circ f^{i-1}\ \forall i \in \mathbb{Z}\},\end{aligned}$$

that is morphisms are commutative diagrams

$$\begin{tikzcd} \cdots \arrow[r, "d_A^{-2}"] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d, "f^{-1}"'] & A^0 \arrow[r, "d_A^0"] \arrow[d, "f^0"'] & A^1 \arrow[r, "d_A^1"] \arrow[d, "f^1"'] & \cdots \\ \cdots \arrow[r, "d_B^{-2}"] & B^{-1} \arrow[r, "d_B^{-1}"] & B^0 \arrow[r, "d_B^0"] & B^1 \arrow[r, "d_B^1"] & \cdots \end{tikzcd}$$

Note that the category $\mathbf{C}(\mathscr{A})$ is also abelian, with kernels and cokernels being calculated position by position.

Remark 17.2. One defines chain complexes in a very similar way, just using lower indices and counting down. This distinction comes from the origins of homological algebra in algebraic topology, where the index often is the dimension of the objects involved. Thus it is natural that the boundary of an n-dimensional object is $n-1$-dimensional (chain complex) and not the other way around (thus called cochain complex).

However in our course a complex is just an abstract sequence of objects and maps, and thus the difference between counting up and counting down is of no concern to us.

Definition 17.3. For a complex $A^\bullet$, and $n \in \mathbb{Z}$, we set

$$\mathtt{B}^n(A^\bullet) = \operatorname{Im} d^{n-1} \text{ and } \mathtt{Z}^n(A^\bullet) = \operatorname{Ker} d^n,$$

called the *n-boundaries* and *n-cycles*, respectively.

Note that since, by definition, $d^n \circ d^{n-1} = 0$, the inclusion $\mathtt{B}^n(A^\bullet) \hookrightarrow A^n$ factors through the inclusion $\mathtt{Z}^n(A^\bullet) \hookrightarrow A^n$. We denote by $\mathtt{H}^n(A^\bullet)$ the cokernel of this map $\mathtt{B}^n(A^\bullet) \hookrightarrow \mathtt{Z}^n(A^\bullet)$, and call it *$n$-th homology* of $A^\bullet$.

Note that all three of these constructions are functorial.

Remark 17.4. In case that our abelian category $\mathscr{A}$ is in fact a category of modules (or any other category where it makes sense to talk about 'elements' of the objects) the above just means that $\mathtt{B}^n(A^\bullet)$ and $\mathtt{Z}^n(A^\bullet)$ are submodules of A^n, such that $\mathtt{B}^n(A^\bullet) \subseteq \mathtt{Z}^n(A^\bullet)$. Now homology is the quotient

$$\mathtt{H}^n(A^\bullet) = \mathtt{Z}^n(A^\bullet)/\mathtt{B}^n(A^\bullet).$$

There are obvious duals to the definition of boundaries, cycles, and homology. (These are *not* what is called coboundaries, cocycles, and cohomology – coboundaries are just the same as boundaries, but distinguishing between counting up and counting down, see Remark 17.2.) However the next lemma tells us that for homology it does not matter if we take this definition or its dual.

Lemma 17.5. *Let $A^\bullet$ be a complex in an abelian category. Then the epimorphism $A^n \twoheadrightarrow \mathtt{B}^{n+1}(A^\bullet)$ factors through the epimorphism $A^n \twoheadrightarrow \operatorname{Cok} d^{n-1}$, and*

$$\mathtt{H}^n(A^\bullet) = \operatorname{Ker}[\operatorname{Cok} d^{n-1} \twoheadrightarrow \mathtt{B}^{n+1}(A^\bullet)].$$

Proof. The factorization follows form $d^n \circ d^{n-1} = 0$.

Now consider the following diagram, where K denotes the kernel of the lemma.

$$\begin{tikzcd}
 & & K \arrow[d, tail] \arrow[dddll, dashed, rounded corners] \\
\mathtt{B}^n(A^\bullet) \arrow[r, hook] \arrow[d, tail] & A^n \arrow[r, two heads] \arrow[d, equal] & \operatorname{Cok} d^{n-1} \arrow[d, two heads] \\
\mathtt{Z}^n(A^\bullet) \arrow[r, hook] \arrow[d, two heads] & A^n \arrow[r, two heads] & \mathtt{B}^{n+1}(A^\bullet) \\
\mathtt{H}^n(A^\bullet) & &
\end{tikzcd}$$

By the snake lemma we have a snake morphism as indicated by the dashed arrow, and it is an isomorphism since both its kernel and cokernel are zero. $\square$

Theorem 17.6 (Long exact sequence of homology)**.** *Let $A^\bullet \rightarrowtail B^\bullet \twoheadrightarrow C^\bullet$ be a short exact sequence in $\mathbf{C}(\mathscr{A})$, for some abelian category $\mathscr{A}$. Then there is a long exact sequence*

$$\cdots \longrightarrow \mathtt{H}^n(A^\bullet) \longrightarrow \mathtt{H}^n(B^\bullet) \longrightarrow \mathtt{H}^n(C^\bullet) \longrightarrow \mathtt{H}^{n+1}(A^\bullet) \longrightarrow \mathtt{H}^{n+1}(B^\bullet) \longrightarrow \cdots .$$

Proof. Note that $\mathtt{Z}^n$ is left exact, and $\operatorname{Cok} d^n$ is right exact. Thus we get exact sequences in the rows of the following commutative diagram.

$$\begin{tikzcd}
 & \operatorname{Cok} d_A^{n-1} \arrow[r] \arrow[d] & \operatorname{Cok} d_B^{n-1} \arrow[r] \arrow[d] & \operatorname{Cok} d_C^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathtt{Z}^{n+1}(A^\bullet) \arrow[r] & \mathtt{Z}^{n+1}(B^\bullet) \arrow[r] & \mathtt{Z}^{n+1}(C^\bullet) &
\end{tikzcd}$$

where the vertical maps are induced by the maps d_A^n, d_B^n, and d_C^n, respectively. In particular they are the composition $\operatorname{Cok} d_A^{n-1} \twoheadrightarrow \mathtt{B}^{n+1}(A^\bullet) \hookrightarrow \mathtt{Z}^{n+1}(A^\bullet)$, and similar for $B^\bullet$ and $C^\bullet$. Thus their cokernels are $\mathtt{H}^{n+1}(A^\bullet)$, $\mathtt{H}^{n+1}(B^\bullet)$, and $\mathtt{H}^{n+1}(C^\bullet)$, respectively. Moreover, by Lemma 17.5, the kernels are $\mathtt{H}^n(A^\bullet)$, $\mathtt{H}^n(B^\bullet)$, and $\mathtt{H}^n(C^\bullet)$, respectively. Now the claim follows from the snake lemma. $\square$

18 Cones and quasi-isomorphisms

In the setup of Theorem 17.6, one easily sees that the maps $\mathtt{H}^n(A^\bullet) \longrightarrow \mathtt{H}^n(B^\bullet)$ and $\mathtt{H}^n(B^\bullet) \longrightarrow \mathtt{H}^n(C^\bullet)$ are just homologies of the original maps in $\mathbf{C}(\mathscr{A})$. However the maps $\mathtt{H}^n(C^\bullet) \longrightarrow \mathtt{H}^{n+1}(A^\bullet)$ are induced by the snake morphism of the snake lemma, and their description is less explicite. We make it explicite here in the special case that all the short exact sequences $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ are split.

Observation 18.1. Let $A^\bullet \rightarrowtail B^\bullet \twoheadrightarrow C^\bullet$ be a short exact sequence in $\mathbf{C}(\mathscr{A})$, such that

$$\forall n \colon B_n = A_n \oplus C_n$$

and the maps are given by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $(0\ 1)$, respectively.

Then d_B^n is given by a 2×2-matrix, say $\left(\begin{smallmatrix} a^n & f^n \\ b^n & c^n \end{smallmatrix}\right)$.

In order for the first map to be a morphism of complexes we need

$$d_B^n \circ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \circ d_A^n,$$

that is $a^n = d_A^n$ and $b^n = 0$. Similarly, in order for the second map to be a morphism of complexes we need $b^n = 0$ and $c^n = d_C^n$.

Finally we require $d_B^n \circ d_B^{n-1} = 0$, with the above that gives

$$0 = \begin{pmatrix} d_A^n & f^n \\ 0 & d_C^n \end{pmatrix} \circ \begin{pmatrix} d_A^{n-1} & f^{n-1} \\ 0 & d_C^{n-1} \end{pmatrix} = \begin{pmatrix} 0 & f^n \circ d_C^{n-1} + d_A^n \circ f^{n-1} \\ 0 & 0 \end{pmatrix}$$

that is $f^n \circ (-d_C^{n-1}) = d_A^n \circ f^{n-1}$.

Conversely we see that any family $(f^n)_{n \in \mathbb{Z}}$ with this property gives rise to a short exact sequence as above.

Definition 18.2 (Shift)**.** Let $A^\bullet$ be a complex. We denote by $A^\bullet[n]$ the complex obtained from $A^\bullet$ by shifting every term n places to the left, that is with

$$(A^\bullet[n])^i = A^{i+n}, \text{ and } d^i_{A^\bullet[n]} = (-1)^n d^{n+i}_{A^\bullet}.$$

Clearly $[n]$ defines an autoequivalence of $\mathbf{C}(\mathscr{A})$, with inverse $[-n]$.

Also note that $\mathtt{H}^i(A^\bullet[n]) = \mathtt{H}^{i+n}(A^\bullet)$.

Definition 18.3 (Cone)**.** Let $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ be a morphism in $\mathbf{C}(\mathscr{A})$. Then the *cone* $\operatorname{Cone}(f^\bullet)$ is the complex

$$\cdots \xrightarrow{\left(\begin{smallmatrix} d_B^{-2} & f^{-1} \\ 0 & -d_A^{-1} \end{smallmatrix}\right)} B^{-1} \oplus A^0 \xrightarrow{\left(\begin{smallmatrix} d_B^{-1} & f^{0} \\ 0 & -d_A^{0} \end{smallmatrix}\right)} B^0 \oplus A^1 \xrightarrow{\left(\begin{smallmatrix} d_B^{0} & f^{1} \\ 0 & -d_A^{1} \end{smallmatrix}\right)} \cdots .$$

By Observation 18.1 above we note that there is a degree-wise split short exact sequence

$$B^\bullet \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \mathrm{Cone}(f^\bullet) \xrightarrow{(0\ 1)} A^\bullet[1].$$

Moreover any degree-wise split short exact sequence is of this form.

Theorem 18.4. *Let* $f^\bullet\colon A^\bullet \longrightarrow B^\bullet$ *be a morphism in* $\mathbf{C}(\mathscr{A})$*. The long exact sequence of homology associated to the short exact sequence*

$$B^\bullet \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \mathrm{Cone}(f^\bullet) \xrightarrow{(0\ 1)} A^\bullet[1]$$

is

$$\cdots \longrightarrow \mathtt{H}^n(A^\bullet) \xrightarrow{\mathtt{H}^n(f^\bullet)} \mathtt{H}^n(B^\bullet) \xrightarrow{\mathtt{H}^n\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \mathtt{H}^n(\mathrm{Cone}(f^\bullet)) \xrightarrow{\mathtt{H}^n(0\ 1)} \mathtt{H}^{n+1}(A^\bullet) \longrightarrow \cdots$$

Proof. The fact that the second and third map are just the homologies of the maps of complexes we started with follows immediately from the construction of the long exact sequence. We need to check that the first map is indeed the n-th homology of the map $f^\bullet$.

We follow the construction. To do so, we consider the following diagram with exact rows, but not columns, where the middle part is just the diagram from the proof of Theorem 17.6.

$$
\begin{tikzcd}
 & & B^n \oplus A^{n+1} \arrow[d, equal] & & \mathtt{Z}^{n+1}(A^\bullet) \arrow[dl, tail] \arrow[dd] & \\
0 \arrow[r] & B^n \arrow[r] \arrow[dd, two heads] & \mathrm{Cone}(f^\bullet)^n \arrow[r] \arrow[dd, two heads] & A^{n+1} \arrow[rr] \arrow[dd, two heads] \arrow[ul, dashed, bend right, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & & 0 \\
 & & & & \mathtt{H}^{n+1}(A^\bullet) \arrow[dl, tail] & \\
 & \mathrm{Cok}\, d_B^{n-1} \arrow[r] \arrow[d] & \mathrm{Cok}\, d_{\mathrm{Cone}(f^\bullet)}^{n-1} \arrow[r] \arrow[d] & \mathrm{Cok}\, d_A^{n} \arrow[rr] \arrow[d] & & 0 \\
0 \arrow[r] & \mathtt{Z}^{n+1}(B^\bullet) \arrow[r] \arrow[dd, tail] \arrow[dl, two heads] & \mathtt{Z}^{n+1}(\mathrm{Cone}(f^\bullet)) \arrow[r] \arrow[dd, tail] & \mathtt{Z}^{n+2}(A^\bullet) \arrow[dd, tail] & & \\
\mathtt{H}^{n+1}(B^\bullet) \arrow[dd, tail] & & & & & \\
0 \arrow[r] & B^{n+1} \arrow[r] \arrow[dl, two heads] & \mathrm{Cone}(f^\bullet)^{n+1} \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend left, "(1\ 0)"] & A^{n+2} \arrow[rr] & & 0 \\
\mathrm{Cok}\, d_B^n & & B^{n+1} \oplus A^{n+2} & & &
\end{tikzcd}
$$

Note that the composition along the columns are just the differentials d_B^n, $\left(\begin{smallmatrix} d_B^n & f^{n+1} \\ 0 & -d_A^{n+1} \end{smallmatrix}\right)$, and $-d_A^{n+1}$, respectively.

Now recall the construction of the snake map from the snake lemma: Using the splitting indicated by the dashed arrows above, we first consider the composition

$$
\mathtt{Z}^{n+1}(A^\bullet) \rightarrowtail A^{n+1} \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} B^n \oplus A^{n+1} \xrightarrow{\left(\begin{smallmatrix} d_B^n & f^{n+1} \\ 0 & -d_A^{n+1} \end{smallmatrix}\right)} B^{n+1} \oplus A^{n+2}
$$
$$
\xrightarrow{(1\ 0)} B^{n+1} \twoheadrightarrow \mathrm{Cok}\, d_B^n.
$$

Multiplying the matrices we see that this is the composition

$$
\mathtt{Z}^{n+1}(A^\bullet) \rightarrowtail A^{n+1} \xrightarrow{f^{n+1}} B^{n+1} \twoheadrightarrow \mathrm{Cok}\, d_B^n.
$$

Thus the induced map on homology is

$$
\mathtt{H}^{n+1}(f^\bullet) \colon \mathtt{H}^{n+1}(A^\bullet) \longrightarrow \mathtt{H}^{n+1}(B^\bullet).
$$

□

Definition 18.5. A morphism $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ in $\mathbf{C}(\mathscr{A})$ is called *quasi-isomorphism* if $\mathtt{H}^n(f^\bullet)$ is an isomorphism for all n.

Corollary 18.6. *Let $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ be a morphism in $\mathbf{C}(\mathscr{A})$. Then $f^\bullet$ is a quasi-isomorphism if and only if the complex* $\mathrm{Cone}(f^\bullet)$ *is exact.*

19 Homotopy

Definition 19.1. A morphism $f^\bullet \colon A^\bullet \longrightarrow B^\bullet$ in $\mathbf{C}(\mathscr{A})$ is called *null-homotopic* if there are morphisms

$$h^n \in \mathrm{Hom}_{\mathscr{A}}(A^n, B^{n-1}) \quad n \in \mathbb{Z}$$

such that

$$\forall n \in \mathbb{Z} \colon f^n = d_B^{n-1} \circ h^n + h^{n+1} \circ d_A^n.$$

Two morphisms $f^\bullet$ and $g^\bullet$ in $\mathrm{Hom}_{\mathbf{C}(\mathscr{A})}(A^\bullet, B^\bullet)$ are called *homotopic* if $f^\bullet - g^\bullet$ is null-homotopic.

Lemma 19.2. *Let $A^\bullet \xrightarrow{e^\bullet} B^\bullet \xrightarrow{f^\bullet} C^\bullet \xrightarrow{g^\bullet} D^\bullet$ be morphisms in $\mathbf{C}(\mathscr{A})$. If $f^\bullet$ is null-homotopic, then so is the composition $g^\bullet \circ f^\bullet \circ e^\bullet$.*

Proof. By definition we have maps h^i such that $f^n = d_C^{n-1} \circ h^n + h^{n+1} \circ d_B^n$. We choose $\widetilde{h}^n = g^{n-1} \circ h^n \circ e^n$. Then

$$\begin{aligned} d_D^{n-1} \circ \widetilde{h}^n + \widetilde{h}^{n+1} \circ d_A^n &= \underbrace{d_D^{n-1} \circ g^{n-1}}_{=g^n \circ d_C^{n-1}} \circ h^n \circ e^n + g^n \circ h^{n+1} \circ \underbrace{e^{n+1} \circ d_A^n}_{=d_B^n \circ e^n} \\ &= g^n \circ (d_C^{n-1} \circ h^n + h^{n+1} \circ d_B^n) \circ e^n \\ &= g^n \circ f^n \circ h^n \end{aligned}$$

□

Definition 19.3. Let $\mathscr{A}$ be an abelian category. The *homotopy category* $\mathbf{K}(\mathscr{A})$ is given by

$$\mathcal{O}b\, \mathbf{K}(\mathscr{A}) = \mathcal{O}b\, \mathbf{C}(\mathscr{A}) \qquad \text{and}$$
$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(A^\bullet, B^\bullet) = \frac{\mathrm{Hom}_{\mathbf{C}(\mathscr{A})}(A^\bullet, B^\bullet)}{\text{homotopy}}$$

that is morphisms are considered the same if their difference is null-homotopic.

Lemma 19.2 shows that this indeed is a category, by making sure that multiplication of morphisms is well-defined.

It follows from the definition that $\mathbf{K}(\mathscr{A})$ inherits the structure of an additive category from $\mathbf{C}(\mathscr{A})$ - the Hom-sets are by definition quotient abelian groups. However $\mathbf{K}(\mathscr{A})$ will typically not be abelian.

Proposition 19.4. *Let $f^\bullet\colon A^\bullet \longrightarrow B^\bullet$ be null-homotopic. Then $\mathtt{H}^n(f^\bullet) = 0$ for all $n \in \mathbb{Z}$.*

In particular the $\mathtt{H}^n$ define functors $\mathbf{K}(\mathscr{A}) \longrightarrow \mathscr{A}$.

Proof. By assumption there are h^n such that $f^n = d_B^{n-1} \circ h^n + h^{n+1} \circ d_A^n$.

First let $\iota_A\colon \mathtt{Z}^n(A^\bullet) \rightarrowtail A^n$, and similar for ι_B. Then $\mathtt{Z}^n(f^\bullet)$ is defined by

$$\iota_B \circ \mathtt{Z}^n(f^\bullet) = f^n \circ \iota_A$$

Inserting the above formula for f^n we obtain that this is equal to

$$d_B^{n-1} \circ h^n \circ \iota_A + h^{n+1} \circ \underbrace{d_A^n \circ \iota_A}_{=0} = d_B^{n-1} \circ h^n \circ \iota_A.$$

Now note that this map clearly factors through $\mathtt{B}^n(B^\bullet) \rightarrowtail B^n$, and thus the induced map on homology vanishes. □

20 Projective and injective resolutions

Definition 20.1. An abelian category $\mathscr{A}$ has *enough projectives* if for any $A \in \mathfrak{Ob}\,\mathscr{A}$ there is an epimorphism $P \twoheadrightarrow A$ from a projective object P to A.

Dually $\mathscr{A}$ has *enough injectives* if for any $A \in \mathfrak{Ob}\,\mathscr{A}$ there is a monomorphism $A \rightarrowtail I$ from A to some injective object I.

Example 20.2. Let R be a ring. The category $\operatorname{Mod} R$ has enough projectives and enough injectives.

Proposition 20.3. *Let $(X, \leqslant)$ be a finite partially ordered set, and $\mathscr{A}$ an abelian category.*

If $\mathscr{A}$ has enough projectives, then so does $\operatorname{presh}_{\mathscr{A}} X$. Dually, if $\mathscr{A}$ has enough injectives, then so does $\operatorname{presh}_{\mathscr{A}} X$.

Proof. We show that $\mathrm{presh}_{\mathscr{A}}\, X$ has enough projectives, the claim about injectives is dual. For a $A \in \mathcal{O}\!\mathit{b}\, \mathscr{A}$ and $i \in X$ we define a presheaf P_i^A by

$$P_i^A(j) = \begin{cases} A & \text{if} j \leqslant i \\ 0 & \text{otherwise.} \end{cases}.$$

One easily sees that

$$\mathrm{Hom}_{\mathrm{presh}_{\mathscr{A}}\, X}(P_i^A, M) = \mathrm{Hom}_{\mathscr{A}}(A, M(i)).$$

Therefore P_i^A is projective provided A is projective in $\mathscr{A}$.

Now let M be an arbitrary $\mathscr{A}$-valued presheaf on X. For $i \in X$, let $A_i \twoheadrightarrow M(i)$ be an epimorphism from a projective object in $\mathscr{A}$. We set $P = \bigoplus_{i \in X} P_i^{A_i}$. Then P is projective, and there is an epimorphism $P \twoheadrightarrow M$ in $\mathrm{presh}_{\mathscr{A}}\, X$. $\square$

Construction 20.4. Let $\mathscr{A}$ be an abelian category with enough projectives, and let $A \in \mathscr{A}$. A *projective resolution* is a complex

$$\cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with projective terms, which is exact, except in postion 0, where $\mathrm{Cok}\, d^{-1} = A$.

Dually, if $\mathscr{A}$ has enough injectives then an *injective resolution* of A is a complex

$$\cdots 0 \longrightarrow 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

with injective terms, which is exact, except in position 0, where $\mathrm{Ker}\, d^0 = A$.

Observation 20.5. Let $\mathscr{A}$ be an abelian category with enough projectives. Then any object $A \in \mathscr{A}$ has a projective resolution. This can be constructed iteratedly: Start with an epimorphism $P^0 \twoheadrightarrow A$, and call A^{-1} its kernel. Given A^i, take an epimorphism $P^i \twoheadrightarrow A^i$, and call A^{i-1} its kernel. Concatenating these short exact sequences we obtain a projective resolution.

Dually, if $\mathscr{A}$ has enough injectives, then any object has an injective resolution.

Construction 20.6. Let A and B be objects in an abelian category having enough projectives. Let $P_A^\bullet$ and $P_B^\bullet$ be projective resolutions of A and B, respectively. Given a morphism $A \xrightarrow{f} B$, we construct (non-canonically) a morphism $P_f^\bullet \colon P_A^\bullet \longrightarrow P_B^\bullet$ such that $\mathtt{H}^0(P_f^\bullet) = f$:

In the diagram below we construct the vertical morphisms from right to left, starting with the given morphism f, such that everything commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & P_A^{-2} \arrow[rr] \arrow[dr, two heads] \arrow[dd, dashed] & & P_A^{-1} \arrow[rr] \arrow[dr, two heads] \arrow[dd, dashed] & & P_A^0 \arrow[dr, two heads] \arrow[dd, dashed] & \\
& & A^{-2} \arrow[ur, tail] \arrow[dd, dashed] & & A^{-1} \arrow[ur, tail] \arrow[dd, dashed] & & A \arrow[dd, "f"] \\
\cdots \arrow[r] & P_B^{-2} \arrow[rr] \arrow[dr, two heads] & & P_B^{-1} \arrow[rr] \arrow[dr, two heads] & & P_B^0 \arrow[dr, two heads] & \\
& & B^{-2} \arrow[ur, tail] & & B^{-1} \arrow[ur, tail] & & B
\end{tikzcd}$$

Here we obtain the morphisms $P_A^{-n} \longrightarrow P_B^{-n}$ using that P_A^{-n} is projective, and thus the composition $P_A^{-n} \twoheadrightarrow A^{-n} \longrightarrow B^{-n}$ may be factored through the epimorphism $P_B^{-n} \twoheadrightarrow B^{-n}$. The morphisms $A^{-n} \longrightarrow B^{-n}$ are kernel morphisms.

Theorem 20.7. *Let $\mathscr{A}$ have enough projectives. Then taking projective resolutions defines a functor*

$$\mathtt{p}\colon \mathscr{A} \longrightarrow \mathbf{K}(\mathscr{A}),$$

such that $\mathtt{H}^0 \circ \mathtt{p} = \mathrm{id}_{\mathscr{A}}$ *and* $\mathtt{H}^n \circ \mathtt{p} = 0$ *for* $n \neq 0$.

Dually, if $\mathscr{A}$ has enough injectives we can define an injective resolution functor

$$\mathtt{i}\colon \mathscr{A} \longrightarrow \mathbf{K}(\mathscr{A}),$$

such that $\mathtt{H}^0 \circ \mathtt{i} = \mathrm{id}_{\mathscr{A}}$ *and* $\mathtt{H}^n \circ \mathtt{i} = 0$ *for* $n \neq 0$.

Proof. We have to show that there is a *unique* map $P_f^\bullet\colon P_A^\bullet \longrightarrow P_B^\bullet$ as in the construction above, for any $f\colon A \longrightarrow B$. Taking differences it is enough to show this for $f = 0$.

So consider the solid part of the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_A^{-2} \arrow[rr] \arrow[dd, "f^{-2}"] & & P_A^{-1} \arrow[rr] \arrow[dd, "f^{-1}"] \arrow[ddll, dashed, "h^{-1}"'] \arrow[dddl, dotted] & & P_A^0 \arrow[dr, two heads] \arrow[dd, "f^0"] \arrow[ddll, dashed, "h^0"'] \arrow[dddl, dotted] & \\
& & & & & & A \arrow[dd, "0"] \\
\cdots \arrow[r] & P_B^{-2} \arrow[rr] \arrow[dr, two heads] & & P_B^{-1} \arrow[rr] \arrow[dr, two heads] & & P_B^0 \arrow[dr, two heads] & \\
& & B^{-2} \arrow[ur, tail] & & B^{-1} \arrow[ur, tail] & & B
\end{tikzcd}$$

Since the composition of f^0 with the epimorphism $P_B^0 \twoheadrightarrow B$ vanishes we see that f^0 factors as indicated by the rightmost dotted map above. Moreover, since P_A^0 is projective, we can lift this dotted map along the epimorphism $P_B^{-1} \twoheadrightarrow B^{-1}$ to obtain a map h^0 as above such that $d_{P_B}^{-1} \circ h^0 = f^0$.

Now observe that $d_{P_B}^{-1} \circ (f^{-1} - h^0 \circ d_{P_A}^{-1}) = d_{P_B}^{-1} \circ f^{-1} - f^0 \circ d_{P_A}^{-1} = 0$. Thus $f^{-1} - h^0 \circ d_{P_A}^{-1}$ factors through the kernel $B^{-2} \rightarrowtail P_B^{-1}$ as indicated by the second dotted arrow. Since P_A^{-1} is projective we may lift this along the epimorphism $P_B^{-2} \twoheadrightarrow B^{-2}$, and obtain a morphism h^{-1} such that $d_{P_B}^{-2} \circ h^{-1} = f^{-1} - h^0 \circ d_{P_A}^{-1}$, or, in other words,

$$f^{-1} = h^0 \circ d_{P_A}^{-1} + d_{P_B}^{-2} \circ h^{-1}.$$

We iterate this construction to obtain a homotopy, thus showing that the map of complexes we started with is in fact null-homotopic. $\square$

Proposition 20.8 (Horseshoe lemma). *Let $A \rightarrowtail B \twoheadrightarrow C$ be a short exact sequence in an abelian category. Assume $P_A^\bullet$ and $P_C^\bullet$ are projective resolutions of A and C, respectively. Then there is a projective resolution $P_B^\bullet$ with $P_B^i = P_A^i \oplus P_C^i$, such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_A^{-2} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P_A^{-1} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P_A^0 \arrow[r, two heads] \arrow[d, tail, "\binom{1}{0}"] & A \arrow[d, tail] \\
\cdots \arrow[r] & P_A^{-2} \oplus P_C^{-2} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & P_A^{-1} \oplus P_C^{-1} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & P_A^0 \oplus P_C^0 \arrow[r, two heads] \arrow[d, two heads, "(0\ 1)"] & B \arrow[d, two heads] \\
\cdots \arrow[r] & P_C^{-2} \arrow[r] & P_C^{-1} \arrow[r] & P_C^0 \arrow[r, two heads] & C
\end{tikzcd}$$

Remark 20.9. In other words, the horseshoe lemma says that $P_B^\bullet$ may be chosen as the cone of a certain map from $P_C^\bullet[-1]$ to $P_A^\bullet$.

Proof. It suffices to consider the first step, and then iterate. Let us denote the given maps by $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$, and $\pi_A \colon P_A^0 \twoheadrightarrow A$ and $\pi_C \colon P_C^0 \twoheadrightarrow C$. Since P_C^0 is projective there is a map $\tilde{\pi_C} \colon P_C^0 \twoheadrightarrow B$ such that $b \circ \tilde{\pi_C} = \pi_C$. It follows that $(b \circ \pi_A\ \tilde{\pi_C})$ is a map $P_A^0 \oplus P_C^0 \longrightarrow B$ making the right part of the diagram above commutative. It follows from the five lemma (Theorem 13.1) that this also is an epimorphism.

Finally note that, by the snake lemma, the kernels also form a short exact sequence, so we may iterate the argument. $\square$

Chapter V

Derived functors

21 Definition and first properties

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ an (additive) functor. Then $\mathtt{F}$ also induces a functor $\mathtt{F}_{\mathbf{K}}\colon \mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{K}(\mathscr{B})$. We use this construction to define derived functors.

Definition 21.1. Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a right exact functor. Assume that $\mathscr{A}$ has enough projectives. Then we define the *n-th left derived functor* of $\mathtt{F}$ by

$$\mathbb{L}_n\mathtt{F} = \mathtt{H}^{-n} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{p}.$$

That is, we take a projective resolution of the object, apply our functor to this projective resolution, and then consider the homology groups of the result.

Dually, if $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ is left exact and $\mathscr{A}$ has enough injectives, we can construct right derived functors as

$$\mathbb{R}^n\mathtt{F} = \mathtt{H}^{n} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{i}.$$

Lemma 21.2. *Let $\mathscr{A}$ have enough projectives, and let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be right exact. Then $\mathbb{L}_0\mathtt{F}$ is naturally isomorphic to $\mathtt{F}$.*

Dually, if $\mathscr{A}$ has enough injectives and the functor is left exact, then $\mathbb{R}^0\mathtt{F} \underset{nat}{\cong} \mathtt{F}$.

Proof. We follow the definition of $\mathbb{L}_0\mathtt{F}$:

Let $A \in \mathcal{O}\!b\,\mathscr{A}$, and let $\cdots \longrightarrow P^{-1} \xrightarrow{d} P^0$ be a projective resolution of A. Then $A = \operatorname{Cok} d$. We apply $\mathtt{F}$, and see that

$$\mathbb{L}_0\mathtt{F}A = \mathtt{H}^0(\cdots \longrightarrow \mathtt{F}P^{-1} \xrightarrow{\mathtt{F}d} \mathtt{F}P_0) = \operatorname{Cok}\mathtt{F}d.$$

But since $\mathtt{F}$ is right exact we have

$$\operatorname{Cok}\mathtt{F}d \cong \mathtt{F}(\operatorname{Cok} d) = \mathtt{F}A. \qquad \square$$

Lemma 21.3. *Let $\mathscr{A}$ have enough projectives, and let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be exact. Then $\mathbb{L}_n\mathtt{F} = 0$ for all non-zero n.*

Dually, if $\mathscr{A}$ has enough injectives and the functor is exact, then $\mathbb{R}^n\mathtt{F} = 0$ except for $n = 0$.

Proof. Since the functor is exact it commutes with taking homology. That is

$$\mathbb{L}_n\mathtt{F} = \mathtt{H}^{-n} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{p} = \mathtt{F} \circ \mathtt{H}^{-n} \circ \mathtt{p} = \begin{cases} \mathtt{F} \circ \mathrm{id}_{\mathscr{A}} & \text{if } n = 0 \\ \mathtt{F} \circ 0 & \text{if } n \neq 0 \end{cases}$$

$\square$

Example 21.4. For the functors Hom and $\otimes$ the derived functors have special names:

$$\begin{aligned} \operatorname{Ext}^n_{\mathscr{A}}(A, -) &= \mathbb{R}^n \operatorname{Hom}_{\mathscr{A}}(A, -) \\ \operatorname{Ext}^n_{\mathscr{A}}(-, A) &= \mathbb{R}^n \operatorname{Hom}_{\mathscr{A}}(-, A) \\ \operatorname{Tor}^R_n(M, -) &= \mathbb{L}_n(M \otimes_R -) \\ \operatorname{Tor}^R_n(-, N) &= \mathbb{L}_n(- \otimes_R N) \end{aligned}$$

In the second line, note that we consider $\operatorname{Hom}_{\mathscr{A}}(-, A)$ as a left exact functor $\mathscr{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$. In particular we calculate the derived functors by taking an injective resolution in $\mathscr{A}^{\mathrm{op}}$, that is a projective resolution in $\mathscr{A}$.

We will see later that $\operatorname{Ext}^n_{\mathscr{A}}(A, -)(B) = \operatorname{Ext}^n_{\mathscr{A}}(-, B)(A)$, and will simply denote this by $\operatorname{Ext}^n_{\mathscr{A}}(A, B)$. (And similar for Tor.)

Example 21.5. We calculate $\operatorname{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/(n), -)(\mathbb{Z}/(m))$:

We start with a projective resolution of $\mathbb{Z}/(m)$. The simplest one is given by $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \longrightarrow 0$.

Now we apply the (non-derived) functor $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} -$, and obtain the complex $0 \longrightarrow \mathbb{Z}/(n) \xrightarrow{\cdot m} \mathbb{Z}/(n) \longrightarrow 0$. The kernel of the non-zero map here is

$$(n/\gcd(m,n))/(n) \cong \mathbb{Z}/(\gcd(m,n)),$$

and the cokernel is also $\mathbb{Z}/(\gcd(m,n))$. Thus

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/(n), -)(\mathbb{Z}/(m)) = \begin{cases} \mathbb{Z}/(\gcd(m,n)) & \text{if } i \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 21.6. *Let $\mathscr{A}$ have enough projectives, and let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be right exact.*

For any short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathscr{A}$ there is a long exact sequence

$$\cdots \longrightarrow \mathbb{L}_2\mathtt{F}C \longrightarrow \mathbb{L}_1\mathtt{F}A \longrightarrow \mathbb{L}_1\mathtt{F}B \longrightarrow \mathbb{L}_1\mathtt{F}C \longrightarrow \mathtt{F}A \longrightarrow \mathtt{F}B \longrightarrow \mathtt{F}C \longrightarrow 0$$

in $\mathscr{B}$.

Proof. By the horseshoe lemma (Proposition 20.8) we may find projective resolutions of A, B, and C fitting into a diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & P_A^{-2} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P_A^{-1} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P_A^{0} \arrow[r, two heads] \arrow[d, tail, "\binom{1}{0}"] & A \arrow[d, tail] \\
\cdots \arrow[r] & P_A^{-2} \oplus P_C^{-2} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & P_A^{-1} \oplus P_C^{-1} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & P_A^{0} \oplus P_C^{0} \arrow[r, two heads] \arrow[d, two heads, "(0\ 1)"] & B \arrow[d, two heads] \\
\cdots \arrow[r] & P_C^{-2} \arrow[r] & P_C^{-1} \arrow[r] & P_C^{0} \arrow[r, two heads] & C
\end{tikzcd}$$

Now, applying $\mathtt{F}$ to these projective resolutions, we obtain the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathtt{F}P_A^{-2} \arrow[r] \arrow[d, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & \mathtt{F}P_A^{-1} \arrow[r] \arrow[d, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & \mathtt{F}P_A^{0} \arrow[d, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\
\cdots \arrow[r] & \mathtt{F}P_A^{-2} \oplus \mathtt{F}P_C^{-2} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & \mathtt{F}P_A^{-1} \oplus \mathtt{F}P_C^{-1} \arrow[r] \arrow[d, two heads, "(0\ 1)"] & \mathtt{F}P_A^{0} \oplus \mathtt{F}P_C^{0} \arrow[d, two heads, "(0\ 1)"] \\
\cdots \arrow[r] & \mathtt{F}P_C^{-2} \arrow[r] & \mathtt{F}P_C^{-1} \arrow[r] & \mathtt{F}P_C^{0}
\end{tikzcd}$$

Note that while $\mathtt{F}$ is not exact, it does preserve direct sums and split short exact sequences.

Now the long exact sequence of the theorem is just the long exact sequence of homology (Theorem 17.6). □

22 Syzygies and dimension shift

Observation 22.1. Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be right exact, and $\mathscr{A}$ have enough projectives.

Then $\mathbb{L}_i\mathtt{F}P = 0$ for all non-zero i and any projective P. (To see this, note that $0 \longrightarrow P \longrightarrow 0$ is a projective resolution.)

The aim of this section is to combine this observation with the long exact sequence of derived functors.

Definition 22.2. Let $\mathscr{A}$ be abelian with enough projectives. For an object A, we construct a *syzygy* of A as the kernel of an epimorphism from a projective object to A, and denote it by ΩA. That is, by definition we have a short exact sequence

$$0 \longrightarrow \Omega A \longrightarrow P \longrightarrow A \longrightarrow 0$$

with P projective.

Remark 22.3. Note that ΩA is not uniquely determined by A: different epimorphisms from projectives may give different syzygies.

In particular Ω is *not* a functor. (It may be applied to morphisms, but this again involves making choices.)

It can be seen that Ω defines an auto-functor of the quotient category

$$\frac{\mathscr{A}}{\text{morphisms factoring through projective objects}}.$$

Definition 22.4. Dually, if $\mathscr{A}$ has enough injectives, we define the *cosyzygy* of an object A to be the cokernel of a monomorphism of A into an injective object. The cosyzygy will be denoted by $\mho A$.

Remark 22.5. It is more usual to denote cosyzygies by Ω^{-1}. However is should be noted that syzygy and cosyzygy are in general not mutually inverse to each other, which this notation seems to suggest.

Theorem 22.6 (Dimension shift)**.** *Let* $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *be right exact, and assume that* $\mathscr{A}$ *has enough projectives. Let* $A \in \mathcal{O}\mathfrak{b}\, \mathscr{A}$. *Then*

$$\mathbb{L}_n\mathtt{F}A = \mathbb{L}_{n-1}\mathtt{F}(\Omega A) \quad \forall n \geqslant 2.$$

Moreover, given a short exact sequence $\Omega A \rightarrowtail P \twoheadrightarrow A$ *with* P *projective, we have*

$$\mathbb{L}_1\mathtt{F}A = \operatorname{Ker}[\mathtt{F}(\Omega A) \longrightarrow \mathtt{F}P].$$

Proof. We consider the short exact sequence $\Omega A \rightarrowtail P \twoheadrightarrow A$, and the long exact sequence of derived functors associated to it. For $n \geqslant 2$ we obtain

$$0 = \mathbb{L}_n\mathtt{F}P \longrightarrow \mathbb{L}_n\mathtt{F}A \longrightarrow \mathbb{L}_{n-1}\mathtt{F}(\Omega A) \longrightarrow \mathbb{L}_{n-1}\mathtt{F}P = 0,$$

and thus the first claim.

For $n = 1$ we have the exact sequence

$$0 = \mathbb{L}_1\mathtt{F}P \longrightarrow \mathbb{L}_1\mathtt{F}A \longrightarrow \mathtt{F}(\Omega A) \longrightarrow \mathtt{F}P,$$

and thus the second claim. □

We also have the dual for left exact functors:

Theorem 22.7. *Let* $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ *be left exact, and assume that* $\mathscr{A}$ *has enough injectives. Let* $A \in \mathcal{O}\mathfrak{b}\, \mathscr{A}$. *Then*

$$\mathbb{R}^n\mathtt{F}A = \mathbb{R}^{n-1}\mathtt{F}(\mho A) \quad \forall n \geqslant 2.$$

Moreover, given a short exact sequence $A \rightarrowtail I \twoheadrightarrow \mho A$ *with* I *injective, we have*

$$\mathbb{R}^1\mathtt{F}A = \operatorname{Cok}[\mathtt{F}I \longrightarrow \mathtt{F}(\mho A)].$$

23 Ext^1 and extensions

Let $\mathscr{A}$ be an abelian category, and A and B objects. We denote by $\mathscr{E}$ the collection of all short exact sequences

$$\mathbb{E}\colon 0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

for some E.

We consider two short exact sequences $\mathbb{E}_1$ and $\mathbb{E}_2$ equivalent if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Note that by the five lemma the map φ necessarily is an isomorphism. Using this fact one may see that the above definition indeed gives rise to an equivalence relation.

Definition 23.1. The *Yoneda-Extension group* is the collection of equivalence classes

$$\mathrm{YExt}^1_{\mathscr{A}}(A, B) = \mathscr{E}/\sim .$$

To explain why this is a group, we first discuss that it is functorial in both A and B:

Construction 23.2. Let $f\colon B_1 \longrightarrow B_2$. Then taking pushouts gives a map $\mathrm{YExt}^1_{\mathscr{A}}(A, B_1) \longrightarrow \mathrm{YExt}^1_{\mathscr{A}}(A, B_2)$, denoted by $f \cdot -$:

$$\begin{tikzcd}
\mathbb{E}\colon & 0 \arrow[r] & B_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
f \cdot \mathbb{E}\colon & 0 \arrow[r] & B_2 \arrow[r] & B_2 \coprod_{B_1} E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Here we use that pushouts of monos are mono, and have the same cokernel.

Dually, if $g\colon A_1 \longrightarrow A_2$, taking pullbacks gives a map

$$- \cdot g\colon \mathrm{YExt}^1_{\mathscr{A}}(A_2, B) \longrightarrow \mathrm{YExt}^1_{\mathscr{A}}(A_1, B).$$

It is possible to see that these constructions commute: $(f \cdot \mathbb{E}) \cdot g = f \cdot (\mathbb{E} \cdot g)$. Hence we may omit brackets in this setup.

Definition 23.3 (Baer sum)**.** Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be in $\mathrm{YExt}^1_{\mathscr{A}}(A, B)$. We first define their coproduct to be

$$\mathbb{E}_1 \oplus \mathbb{E}_2\colon 0 \longrightarrow B \oplus B \longrightarrow E_1 \oplus E_2 \longrightarrow A \oplus A \longrightarrow 0 \in \mathrm{YExt}_{\mathscr{A}}(A \oplus A, B \oplus B),$$

where all maps are diagonal.

Now the *Baer sum* of $\mathbb{E}_1$ and $\mathbb{E}_2$ is

$$\mathbb{E}_1 + \mathbb{E}_2 = (1\ 1) \circ (\mathbb{E}_1 \oplus \mathbb{E}_2) \circ \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \in \mathrm{YExt}^1_{\mathscr{A}}(A, B).$$

Theorem 23.4. *The Yoneda-Ext of two objects, together with Baer sum, forms an abelian group (provided it is a set). The zero-element of this abelian group is given by the split short exact sequence.*

This group structure turns $\mathrm{YExt}^1_{\mathscr{A}}$ *into an additive functor* $\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathbf{Ab}$.

Proof. It is clear from the construction that the Baer sum is commutative.

For $\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \in \mathrm{YExt}^1_{\mathscr{A}}(A, B)$ one may see that

$$\mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3 = (1\ 1\ 1) \circ (\mathbb{E}_1 \oplus \mathbb{E}_2) \circ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

independent of brackets, that is Baer sum is associative.

Next we observe that for any short exact sequence $\mathbb{E}$ we have that both $0 \cdot \mathbb{E}$ and $\mathbb{E} \cdot 0$ are split short exact. Indeed we have the following pushout diagram

$$\begin{tikzcd}
\mathbb{E}\colon & 0 \arrow[r] & B_1 \arrow[r] \arrow[d, "0"] & E \arrow[r, "\pi"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ \pi \end{smallmatrix}\right)"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \cdot \mathbb{E} & 0 \arrow[r] & B_2 \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & B_2 \oplus A \arrow[r, "(0\ 1)"] & A \arrow[r] & 0
\end{tikzcd}$$

(and a similar one for the case of pullbacks along zero-morphisms).

Now we check that for two maps f and g from B_1 to B_2, and an extension $\mathbb{E} \in \mathrm{YExt}^1_{\mathscr{A}}(A, B_1)$, we have $(f+g) \cdot \mathbb{E} = f \cdot \mathbb{E} + g \cdot \mathbb{E}$. As a first step, consider the commutative diagram

$$\begin{tikzcd}
\mathbb{E}\colon & 0 \arrow[r] & B_1 \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & E \arrow[r] \arrow[d] \arrow[dd, bend left, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & A \arrow[r] \arrow[d, equal] & 0 \\
\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \cdot \mathbb{E}\colon & 0 \arrow[r] & B_1 \oplus B_1 \arrow[r] \arrow[d, equal] & \widetilde{E} \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & 0 \\
\mathbb{E} \oplus \mathbb{E}\colon & 0 \arrow[r] & B_1 \oplus B_1 \arrow[r] & E \oplus E \arrow[r] & A \oplus A \arrow[r] & 0
\end{tikzcd}$$

where the dashed arrow exists by the pushout property of the upper left square. We see that

$$\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \cdot \mathbb{E} = (\mathbb{E} \oplus \mathbb{E}) \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right).$$

Now we calculate

$$\begin{aligned}
f \cdot \mathbb{E} + g \cdot \mathbb{E} &= (1\ 1)(f \cdot \mathbb{E} \oplus g \cdot \mathbb{E}) \circ \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\
&= (f\ g) \cdot (\mathbb{E} \oplus \mathbb{E}) \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \\
&= (f\ g) \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \cdot \mathbb{E} \\
&= (f+g) \cdot \mathbb{E}
\end{aligned}$$

Finally we use the above observations to verify that the split exact sequences are a neutral element, and that there are inverses:

Let $\mathbb{E} \in \mathrm{YExt}_{\mathscr{A}}(A, B)$, and let $\mathbb{E}_{\mathrm{split}}$ denote the split exact sequence between the same two objects. Then

$$\mathbb{E} + \mathbb{E}_{\mathrm{split}} = 1 \cdot \mathbb{E} + 0 \cdot \mathbb{E} = (1+0) \cdot \mathbb{E} = \mathbb{E}.$$

Similarly we check that $(-1) \cdot \mathbb{E}$ is an inverse of $\mathbb{E}$:

$$\mathbb{E} + (-1) \cdot \mathbb{E} = (1-1) \cdot \mathbb{E} = 0 \cdot \mathbb{E}$$

is the split exact sequence. □

Theorem 23.5. *Assume $\mathscr{A}$ has enough projectives. Then*

$$\mathrm{YExt}^1_{\mathscr{A}}(A, B) = \mathrm{Ext}^1_{\mathscr{A}}(-, B)(A).$$

Dually, if $\mathscr{A}$ has enough injectives then

$$\mathrm{YExt}^1_{\mathscr{A}}(A,B) = \mathrm{Ext}^1_{\mathscr{A}}(A,-)(B).$$

In particular if $\mathscr{A}$ has both enough projectives and enough injectives then

$$\mathrm{Ext}^1_{\mathscr{A}}(-,B)(A) = \mathrm{Ext}^1_{\mathscr{A}}(A,-)(B).$$

Proof. We prove the first claim. The second one is dual, and the third one then follows immediately.

Consider a short exact sequence

$$\mathbb{E}_{\mathrm{p}}\colon 0 \longrightarrow \Omega A \xrightarrow{\iota} P \xrightarrow{\pi} A \longrightarrow 0$$

with P projective. We may consider this an element of $\mathrm{YExt}^1_{\mathscr{A}}(A,\Omega A)$.

Now multiplication with $\mathbb{E}_{\mathrm{p}}$ gives a map

$$- \cdot \mathbb{E}_{\mathrm{p}}\colon \mathrm{Hom}_{\mathscr{A}}(\Omega A, B) \longrightarrow \mathrm{YExt}^1_{\mathscr{A}}(A,B).$$

We claim that this map is surjective, and that its kernel consist precisely of the morphisms factoring through ι.

To see surjectivity, consider the following diagram for any $\mathbb{E} \in \mathrm{YExt}^1_{\mathscr{A}}(A,B)$:

$$\begin{tikzcd}
\mathbb{E}_{\mathrm{p}}\colon & 0 \arrow[r] & \Omega A \arrow[r] \arrow[d, dashed, "f"'] & P \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}\colon & 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Here the right dashed arrow exists by the lifting property of projectives, and the left dashed arrow is a kernel morphism. It follows from the characterization of pushouts (see Theorem 13.2) that the left square is a pushout, that is that $\mathbb{E} = f \cdot \mathbb{E}_{\mathrm{p}}$ for the morphism f found in the diagram.

To determine the kernel of the map $- \cdot \mathbb{E}_{\mathrm{p}}$, note that a morphism f is in the kernel if and only if we can find a commutative diagram

$$\begin{tikzcd}
\mathbb{E}_{\mathrm{p}}\colon & 0 \arrow[r] & \Omega A \arrow[r, "\iota"] \arrow[d, "f"'] & P \arrow[r, "\pi"] \arrow[d, dashed, "\binom{r}{s}"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}\colon & 0 \arrow[r] & B \arrow[r, "\binom{1}{0}"] & B \oplus A \arrow[r, "(0\ 1)"] & A \arrow[r] & 0
\end{tikzcd}$$

By commutativity of the right square we need $s = \pi$, and then the left square commutes if and only if $r \circ \iota = f$. It follows that the kernel consists precisely of the maps factoring through ι.

But, by dimension-shift (see Theorem 22.7) we have

$$\mathrm{Ext}^1_{\mathscr{A}}(-, B)(A) = \mathrm{Cok}[\mathrm{Hom}_{\mathscr{A}}(P, B) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(\Omega A, B)],$$

that is $\mathrm{Ext}^1_{\mathscr{A}}(-, B)(A)$ is also the quotient of $\mathrm{Hom}_{\mathscr{A}}(\Omega A, B)$ modulo morphisms factoring through ι. □

24 Total complexes - balancing Tor and Ext

Definition 24.1. A *double complex* is an infinite commutative square pattern

$$\begin{tikzcd}
 & \vdots \arrow[d, "d_{\mathrm{v}}^{m-1,n-2}"] & \vdots \arrow[d, "d_{\mathrm{v}}^{m,n-1}"] & \vdots \arrow[d, "d_{\mathrm{v}}^{m+1,n-1}"] & \\
\cdots \arrow[r, "d_{\mathrm{h}}^{m-2,n-1}"] & X^{m-1,n-1} \arrow[r, "d_{\mathrm{h}}^{m-1,n-1}"] \arrow[d, "d_{\mathrm{v}}^{m-1,n-1}"] & X^{m,n-1} \arrow[r, "d_{\mathrm{h}}^{m,n-1}"] \arrow[d, "d_{\mathrm{v}}^{m,n-1}"] & X^{m+1,n-1} \arrow[r, "d_{\mathrm{h}}^{m+1,n-1}"] \arrow[d, "d_{\mathrm{v}}^{m+1,n-1}"] & \cdots \\
\cdots \arrow[r, "d_{\mathrm{h}}^{m-2,n}"] & X^{m-1,n} \arrow[r, "d_{\mathrm{h}}^{m-1,n}"] \arrow[d, "d_{\mathrm{v}}^{m-1,n}"] & X^{m,n} \arrow[r, "d_{\mathrm{h}}^{m,n}"] \arrow[d, "d_{\mathrm{v}}^{m,n}"] & X^{m+1,n} \arrow[r, "d_{\mathrm{h}}^{m+1,n}"] \arrow[d, "d_{\mathrm{v}}^{m+1,n}"] & \cdots \\
\cdots \arrow[r, "d_{\mathrm{h}}^{m-2,n+1}"] & X^{m-1,n+1} \arrow[r, "d_{\mathrm{h}}^{m-1,n+1}"] \arrow[d, "d_{\mathrm{v}}^{m-1,n+1}"] & X^{m,n+1} \arrow[r, "d_{\mathrm{h}}^{m,n+1}"] \arrow[d, "d_{\mathrm{v}}^{m,n+1}"] & X^{m+1,n+1} \arrow[r, "d_{\mathrm{h}}^{m+1,n+1}"] \arrow[d, "d_{\mathrm{v}}^{m+1,n+1}"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

such that the composition of any two vertical or any two horizontal morphisms vanishes.

In other words, a double complex is just an object in the category $\mathbf{C}(\mathbf{C}(\mathscr{A}))$.

Definition 24.2. Let $X^{\bullet,\bullet}$ be a double complex, and assume that for any s, the infinite coproduct $\coprod_{m\in\mathbb{Z}} X^{m,s-m}$ exists. (For instance this is true if on every diagonal there are only finitely many non-zero objects.)

Then the *total complex* of $X^{\bullet,\bullet}$ is given by

$$\mathrm{Tot}(X^{\bullet,\bullet})^s = \coprod_{m\in\mathbb{Z}} X^{m,s-m},$$

with the differential given on components by

$$X^{m,s-m} \longrightarrow X^{m',s+1-m'} : \begin{cases} d_{\mathrm{v}}^{m,s-m} & \text{if } m' = m \\ (-1)^{s-m} d_{\mathrm{h}}^{m,s-m} & \text{if } m' = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 24.3. Note that cones are a special case of total complexes, where the only non-zero objects lie in rows -1 and 0.

Proposition 24.4. *Let $X^{\bullet,\bullet}$ be a double complex concentrated in finitely many rows. (That is there are $a \leqslant b$ such that $X^{m,n} = 0$ whenever $n < a$ or $n > b$.) Assume that all rows of $X^{\bullet,\bullet}$ are exact. Then the total complex $\mathrm{Tot}(X^{\bullet,\bullet})$ is exact.*

Proof. Let

$$Y^{m,n} = \begin{cases} X^{m,n} & \text{if } n > a \\ 0 & \text{if } n \leqslant a \end{cases}$$

that is $Y^{\bullet,\bullet}$ is obtained from $X^{\bullet,\bullet}$ by removing the top non-zero row.

Then one may observe that there is a natural map

$$f^\bullet : X^{\bullet,a}[-a-1] \longrightarrow \mathrm{Tot}(Y^{\bullet,\bullet})$$

and

$$\mathrm{Tot}(X^{\bullet,\bullet}) = \mathrm{Cone}(f^\bullet).$$

Now we may assume inductively that $\mathrm{Tot}(Y^{\bullet,\bullet})$ is exact, and it then follows from the long exact sequence of homology that also all homologies of $\mathrm{Tot}(X^{\bullet,\bullet})$ vanish. □

Corollary 24.5. *Let $X^{\bullet,\bullet}$ be a double complex such that all diagonals are finite. (That is for any s there are only finitely many m such that $X^{m,s-m} \neq 0$.) Assume all rows of $X^{\bullet,\bullet}$ are exact. Then the total complex $\mathrm{Tot}(X^{\bullet,\bullet})$ is exact.*

Proof. For any given position, we may disregard the rows of $X^{\bullet,\bullet}$ such that $X^{s-n,n} = 0$. Hence exactness in position s follows from Proposition 24.4 above. Since this applies to any given position the entire complex is exact. □

Theorem 24.6 (Balancing Ext)**.** *Let $\mathscr{A}$ be an abelian category with enough projectives and enough injectives. Then for any $A, B \in \mathfrak{Ob}\,\mathscr{A}$*

$$\mathrm{Ext}^n_{\mathscr{A}}(A,-)(B) = \mathrm{Ext}^n_{\mathscr{A}}(-,B)(A).$$

Proof. We choose a projective resolution $P^\bullet$ of A, and an injective resolution $I^\bullet$ of B.

Recall that the two Ext-groups of the theorem are by definition the homologies of $\mathrm{Hom}_{\mathscr{A}}(A, I^\bullet)$ and $\mathrm{Hom}_{\mathscr{A}}(P^\bullet, B)$, respectively. We will connect these two complexes via the third complex $\mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet))$, showing that there are two quasi-isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(P^\bullet, B) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet)) \longleftarrow \mathrm{Hom}_{\mathscr{A}}(A, I^\bullet).$$

It then follows immediately that all three complexes have the same homologies.

We denote the exact complex

$$\cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

by $\overline{P}^\bullet$, and similarly the exact complex

$$\cdots 0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

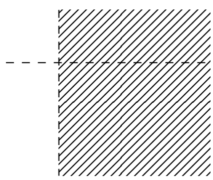
by $\overline{I}^\bullet$.

We consider the double complex $\mathrm{Hom}_{\mathscr{A}}(\overline{P}^\bullet, \overline{I}^\bullet)$, and its versions with $P^\bullet$ instead of $\overline{P}^\bullet$ and $I^\bullet$ instead of $\overline{I}^\bullet$. (In the following picture we write (X, Y)

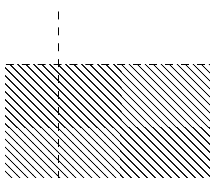
for $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ to save space.)

$$
\begin{tikzcd}
(A,B) \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] & (P^0,B) \arrow[r] \arrow[d, rightarrowtail] & (P^1,B) \arrow[r] \arrow[d, rightarrowtail] & (P^2,B) \arrow[d, rightarrowtail] \\
(A,I^0) \arrow[r, rightarrowtail] \arrow[d] & (P^0,I^0) \arrow[r] \arrow[d] & (P^1,I^0) \arrow[r] \arrow[d] & (P^2,I^0) \arrow[d] \\
(A,I^1) \arrow[r, rightarrowtail] \arrow[d] & (P^0,I^1) \arrow[r] \arrow[d] & (P^1,I^1) \arrow[r] \arrow[d] & (P^2,I^1) \arrow[d] \\
(A,I^2) \arrow[r, rightarrowtail] & (P^0,I^2) \arrow[r] & (P^1,I^2) \arrow[r] & (P^2,I^2)
\end{tikzcd}
$$

We note that

$\mathrm{Hom}_{\mathscr{A}}(P^\bullet, \overline{I}^\bullet) =$  exact columns $\Longrightarrow$ exact total complex

and

$\mathrm{Hom}_{\mathscr{A}}(\overline{P}^\bullet, I^\bullet) =$  exact rows $\Longrightarrow$ exact total complex

On the other hand there is the morphism $\mathrm{Hom}_{\mathscr{A}}(P^\bullet, B) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet))$ (essentially given by the morphisms crossing the vertical dashed line above), whose cone is $\mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, \overline{I}^\bullet))$. In particular the cone is exact, so the morphism is a quasi-isomorphism.

Similarly the natural morphism $\mathrm{Hom}_{\mathscr{A}}(A, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet))$ is a quasi-isomorphism, since its cone $\mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(\overline{P}^\bullet, I^\bullet)$ is exact.

Now the two quasi-isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(P^\bullet, B) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet)) \longleftarrow \mathrm{Hom}_{\mathscr{A}}(A, I^\bullet)$$

give rise to isomorphisms

$$\mathrm{H}^{-n}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, B)) \cong \mathrm{H}^{-n}(\mathrm{Tot}(\mathrm{Hom}_{\mathscr{A}}(P^\bullet, I^\bullet))) \cong \mathrm{H}^{-n}(\mathrm{Hom}_{\mathscr{A}}(A, I^\bullet)).$$

Now note that the left hand term is by definition $\mathrm{Ext}^n_{\mathscr{A}}(-, B)(A)$, while the right hand term is $\mathrm{Ext}^n_{\mathscr{A}}(A, -)(B)$. □

Theorem 24.7 (Balancing Tor). *Let R be ring, M a right and N a left R-module. Then*

$$\mathrm{Tor}^R_n(M, -)(N) = \mathrm{Tor}^R_n(-, N)(M).$$

Proof. The proof is very similar to the proof of Theorem 24.6 above. Here we start with two projective resolutions $P^\bullet_M$ and $P^\bullet_N$ of M and N respectively. We then proceed as before to show that we have quasi-isomorphisms

$$M \otimes_R P^\bullet_N \longrightarrow \mathrm{Tot}(P^\bullet_M \otimes_R P^\bullet_N) \longleftarrow P^\bullet_M \otimes_R N.$$

It then follows that the homologies of all three complexes coincide. □

25 Small global dimension

Throughout this section, let $\mathscr{A}$ be an abelian category that has enough projectives or enough injectives.

Definition 25.1. The *global dimension* of $\mathscr{A}$ is

$$\mathrm{gl.dim}\,\mathscr{A} = \sup\{n \in \mathbb{N}_0 \mid \exists A, B \in \mathcal{O}b\,\mathscr{A} \colon \mathrm{Ext}^n_{\mathscr{A}}(A, B) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}.$$

An abelian category is called

- *semisimple* if $\mathrm{gl.dim}\,\mathscr{A} = 0$;
- *hereditary* if $\mathrm{gl.dim}\,\mathscr{A} \leqslant 1$.

Proposition 25.2. *The following are equivalent:*

(1) $\mathscr{A}$ is semisimple;

(2) all objects in $\mathscr{A}$ are projective;

(3) all objects in $\mathscr{A}$ are injective;

(4) all epimorphisms in $\mathscr{A}$ are split epimorphisms;

(5) all monomorphisms in $\mathscr{A}$ are split monomorphisms.

Proof. We show (1) $\Longrightarrow$ (5) $\Longrightarrow$ (3) $\Longrightarrow$ (1). The proof of (1) $\Longrightarrow$ (4) $\Longrightarrow$ (2) $\Longrightarrow$ (1) is similar.

Assume first that $\mathscr{A}$ is semisimple. Then $\mathrm{YExt}^1_{\mathscr{A}} = 0$, hence all short exact sequences split. In particular any monomorphism splits.

If any monomorphism splits then the lifting property for injectives is automatic, so all objects are injective.

Finally, if all objects are injective, then $\mathrm{Ext}^n_{\mathscr{A}}(A, -)(B) = 0$ for all $n > 0$. (Note that B is its own injective resolution.) $\square$

Example 25.3. Let $\mathbb{F}$ be a field. Then both $\mathrm{Mod}\,\mathbb{F}$ and $\mathrm{mod}\,\mathbb{F}$ are semisimple abelian categories.

Definition 25.4. Assume $\mathscr{A}$ has enough projectives. Then the *projective dimension* $\mathrm{pd}\,A$ of an object A is the smallest n, such that A has a projective resolution of the form

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots .$$

(We say $\mathrm{pd}\,A = \infty$ if all projective resolutions of A are infinite.)

Dually, if $\mathscr{A}$ has enough injectives, then the *injective dimension* $\mathrm{id}\,A$ of an object A is the smallest n such that A has an injective resolution

$$\cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0 \longrightarrow \cdots .$$

Remark 25.5. Clearly an object A is projective if and only if $\mathrm{pd}\,A = 0$, and injective if and only if $\mathrm{id}\,A = 0$.

Theorem 25.6. *Assume $\mathscr{A}$ has enough projectives, and let $A \in \mathrm{Ob}\,\mathscr{A}$. Then*

$$\mathrm{pd}\,A = \sup\{n \in \mathbb{N}_0 \mid \exists B \in \mathrm{Ob}\,\mathscr{A} \colon \mathrm{Ext}^n_{\mathscr{A}}(A, B) \neq 0\}.$$

Proof. If $\operatorname{pd} A = n$ then $\operatorname{Ext}^i_{\mathscr{A}}(A, B) = 0$ for any $i > n$, and thus we have the inequality $\geqslant$.

Assume now that $\operatorname{Ext}^i_{\mathscr{A}}(A, -) = 0$ for some i. By dimension shift it follows that $\operatorname{Ext}^1_{\mathscr{A}}(\Omega^{i-1} A, -) = 0$. Interpreting this as Yoneda-Ext, we see that any epimorphism to $\Omega^{i-1} A$ splits, that is that $\Omega^{i-1} A$ is projective.

Now we have a projective resolution of length $i - 1$, given by

$$0 \longrightarrow \Omega^{i-1} A \longrightarrow P^{2-i} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow 0$$

showing that $\operatorname{pd} A \leqslant i - 1$. □

We also have the dual of the above theorem:

Theorem 25.7. *Assume $\mathscr{A}$ has enough injectives, and let $A \in \mathfrak{Ob}\, \mathscr{A}$. Then*

$$\operatorname{id} A = \sup\{n \in \mathbb{N}_0 \mid \exists B \in \mathfrak{Ob}\, \mathscr{A} \colon \operatorname{Ext}^n_{\mathscr{A}}(B, A) \neq 0\}.$$

Corollary 25.8. *Assume $\mathscr{A}$ has enough projectives. Then*

$$\operatorname{gl.dim} \mathscr{A} = \sup\{\operatorname{pd} A \mid A \in \mathfrak{Ob}\, \mathscr{A}\}.$$

Dually, if $\mathscr{A}$ has enough injectives, then

$$\operatorname{gl.dim} \mathscr{A} = \sup\{\operatorname{id} A \mid A \in \mathfrak{Ob}\, \mathscr{A}\}.$$

Proposition 25.9. *Assume $\mathscr{A}$ has enough projectives. Then $\mathscr{A}$ is hereditary if and only if all subobjects of projective objects are projective.*

Remark 25.10. This explains the name "hereditary": subobjects inherit the property of being projective.

Proof. Assume first that any subobject of a projective is projective. Then it follows that any object has a projective resolution with at most two non-zero terms. Thus $\mathscr{A}$ is hereditary by Corollary 25.8.

Assume conversely that $\mathscr{A}$ is hereditary, and let $A \rightarrowtail P$ be a subobject of a projective. We denote by P/A the cokernel of this inclusion, and observe that

$$\operatorname{Ext}^1_{\mathscr{A}}(A, -) = \operatorname{Ext}^2_{\mathscr{A}}(P/A, -) = 0$$

where the first equality is dimension shift, and the latter comes from the definition of hereditary. It follows that A is projective. □

Theorem 25.11. *Let R be right noetherian. Then the category* $\operatorname{mod} R$ *of finitely generated right R-modules is hereditary if and only if all right ideals of R are projective.*

Proof. "only if" is clear, since right ideals are submodules of the projective module R.

"if": It suffices to show that any submodule of R^n is projective (since all projective objects are direct summands of free modules). We show this by induction on n, the case $n = 1$ holding by assumption.

Let M be a submodule of R^n, and consider the split short exact sequence

$$0 \longrightarrow R \longrightarrow \underbrace{R \oplus R^{n-1}}_{\cong R^n} \longrightarrow R^{n-1} \longrightarrow 0.$$

We denote by I and K the image and kernel of the composition $M \longrightarrow R^{n-1}$. Thus we have the following commutative diagram

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, tail, dashed] & M \arrow[r, two heads] \arrow[d, tail] & I \arrow[d, tail] \\ R \arrow[r, tail] & R^n \arrow[r, two heads] & R^{n-1} \end{tikzcd}$$

where the dashed map is the kernel morphism, and it is mono by commutativity of the left square.

Now inductively both K and I are projective, hence the upper short exact sequence splits, and $M \cong K \oplus I$ also is projective. $\square$

Remark 25.12. More generally, one can show that the category $\operatorname{Mod} R$ is hereditary if and only if all right ideals of R are projective.

Example 25.13. Let R be a principal ideal domain. (That is a commutative ring without zero-divisors, such that every ideal is generated by a single element.) Then $\operatorname{Mod} R$ is hereditary.

In particular $\operatorname{Mod} \mathbb{Z}$ is hereditary, and for any field $\mathbb{F}$ the category of modules over the polynomial ring $\operatorname{Mod} \mathbb{F}[X]$ is hereditary.

Remark 25.14. One can show that for a field $\mathbb{F}$, one has

$$\operatorname{gl.dim} \mathbb{F}[X_1, \ldots, X_d] = d.$$

Chapter VI

Triangulated categories

26 Motivation – triangles in the homotopy category

Throughout this section, let $\mathscr{A}$ be an abelian category. We have seen that for a morphism of complexes $f^\bullet$

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow \mathrm{Cone}(f^\bullet) \longrightarrow A^\bullet[1]$$

is a complex in $\mathbf{K}(\mathscr{A})$, giving rise to a long exact sequence of homology.

Now we take a different point of view, and say we consider the infinite complex

$$\begin{aligned} \cdots \longrightarrow A^\bullet[n] \longrightarrow B^\bullet[n] \longrightarrow \mathrm{Cone}(f^\bullet)[n] \\ \longrightarrow A^\bullet[n+1] \longrightarrow B^\bullet[n+1] \longrightarrow \mathrm{Cone}(f^\bullet)[n+1] \longrightarrow \cdots \end{aligned}$$

in the homotopy category. Since this complex is (up to shift) 3-periodic, we denote it by the triangle of objects and morphisms

$$\begin{tikzcd}
A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\
& \mathrm{Cone}(f^\bullet) \arrow[ul, "|" marking] &
\end{tikzcd}$$

(where the decorated arrow indicates that this represents a morphism to $A^\bullet[1]$).

The next result shows that, while the triangle is defined starting with $f^\bullet$, it has no preferred side.

Proposition 26.1. *Let*

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\iota^\bullet} \mathrm{Cone}(f^\bullet) \xrightarrow{\pi^\bullet} A^\bullet[1]$$

be a triangle as above, that is $\iota^n = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ *and* $\pi^n = (0\ 1)$ *for all* n.

Then there is an isomorphism $\varphi^\bullet\colon \mathrm{Cone}(\iota^\bullet) \longrightarrow A^\bullet[1]$ *in* $\mathbf{K}(\mathscr{A})$ *such that the following diagram commutes.*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\iota^\bullet"] \arrow[d, equal] & \mathrm{Cone}(f^\bullet) \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, equal] & \mathrm{Cone}(\iota^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, "\varphi^\bullet"] & B^\bullet[1] \arrow[d, equal] \\
B^\bullet \arrow[r, "\iota^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "\pi^\bullet"] & A^\bullet[1] \arrow[r, "-f^\bullet[1]"] & B^\bullet[1]
\end{tikzcd}$$

Proof. We start by calculating that

$$\mathrm{Cone}(\iota^\bullet)^n = \mathrm{Cone}(f^\bullet)^n \oplus B^{n+1} = B^n \oplus A^{n+1} \oplus B^{n+1}$$

and

$$d^n_{\mathrm{Cone}(\iota^\bullet)} = \begin{pmatrix} d_{\mathrm{Cone}(f^\bullet)} & \iota^{n+1} \\ 0 & -d_B^{n+1} \end{pmatrix} = \begin{pmatrix} d_B^n & f^{n+1} & 1 \\ 0 & -d_A^{n+1} & 0 \\ 0 & 0 & -d_B^{n+1} \end{pmatrix}.$$

We consider the morphisms $\varphi^\bullet\colon \mathrm{Cone}(\iota^\bullet) \longrightarrow A^\bullet[1]$ given by $\varphi^n = (0\ 1\ 0)$ and $\psi^\bullet\colon A^\bullet[1] \longrightarrow \mathrm{Cone}(\iota^\bullet)$ given by $\psi^n = \begin{pmatrix} 0 \\ 1 \\ -f^{n+1} \end{pmatrix}$. A straightforward calculation shows that these are indeed morphisms of complexes.

Now we check the following three claims:

(1) The square

$$\begin{tikzcd}
\mathrm{Cone}(f^\bullet) \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, equal] & \mathrm{Cone}(\iota^\bullet) \arrow[d, "\varphi^\bullet"] \\
\mathrm{Cone}(f^\bullet) \arrow[r, "\pi^\bullet"] & A^\bullet[1]
\end{tikzcd}$$

commutes in the category $\mathbf{C}(\mathscr{A})$.

(2) The square

$$\begin{tikzcd} \operatorname{Cone}(\iota^\bullet) \arrow[r, "(0\;1)"] & B^\bullet[1] \\ A^\bullet[1] \arrow[u, "\psi^\bullet"] \arrow[r, "-f^\bullet[1]"] & B^\bullet[1] \arrow[u, equal] \end{tikzcd}$$

commutes in the category $\mathbf{C}(\mathscr{A})$.

(3) $\varphi^\bullet \circ \psi^\bullet = 1$ in the category $\mathbf{C}(\mathscr{A})$.

(4) $1 - \psi^\bullet \circ \varphi^\bullet$ is null homotopic.

(1), (2), and (3) are straightforward matrix calculations, which are left to the reader. We only check (4) here. First we calculate

$$1 - \psi^n \circ \varphi^n = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -f^{n+1} \end{pmatrix} \circ (0\;1\;0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^{n+1} & 1 \end{pmatrix}.$$

Now we set $h^n = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} : \operatorname{Cone}(\iota^\bullet)^n \longrightarrow \operatorname{Cone}(\iota^\bullet)^{n-1}$ and see that

$$\begin{aligned} & d^{n-1}_{\operatorname{Cone}(\iota^\bullet)} \circ h^n + h^{n+1} \circ d^n_{\operatorname{Cone}(\iota^\bullet)} \\ &= \begin{pmatrix} d^n_B & f^{n+1} & 1 \\ 0 & -d^{n+1}_A & 0 \\ 0 & 0 & -d^{n+1}_B \end{pmatrix} \circ \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} \circ \begin{pmatrix} d^n_B & f^{n+1} & 1 \\ 0 & -d^{n+1}_A & 0 \\ 0 & 0 & -d^{n+1}_B \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -d^n_B & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d^n_B & f^{n+1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & f^{n+1} & 1 \end{pmatrix} \end{aligned}$$

Now note that (3) and (4) together imply that $\varphi^\bullet$ and $\psi^\bullet$ are mutually inverse isomorphisms in $\mathbf{K}(\mathscr{A})$, and thus (2) implies that also the rightmost square in the proposition commutes up to homotopy. ☐

27 Definition

Definition 27.1. A *triangulated category* is an additive category $\mathscr{T}$, together with an autoequivalence $[1]\colon \mathscr{T} \longrightarrow \mathscr{T}$, and a class Δ of diagrams of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ such that

(T1)
- For any morphism $f\colon X \longrightarrow Y$ in $\mathscr{T}$, there is a diagram

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

in Δ.

- For any object X, the diagram $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$ is in Δ.
- Δ is closed under isomorphism.

(T2) For any diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in Δ also the diagrams

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \text{ and}$$
$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are in Δ.

(T3) Given the solid part of a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, dashed, "w"'] & X[1] \arrow[d, "{u[1]}"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

where the square commutes, and the rows are in Δ, one can always find a morphism w as indicated above such that the entire diagram becomes commutative.

(T4) *Octahedral axiom*: Given the solid part of the following diagram, where the two rows and the left column are in Δ,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
 & X' \arrow[r, equal] \arrow[d, "h"'] & X' \arrow[r, "h"] \arrow[d] & Y[1] \\
 & Y[1] \arrow[r, "g[1]"] & Z'[1] &
\end{tikzcd}$$

there are morphisms as indicated by the dashed arrows, such that also the second column is in Δ, and the entire diagram commutes.

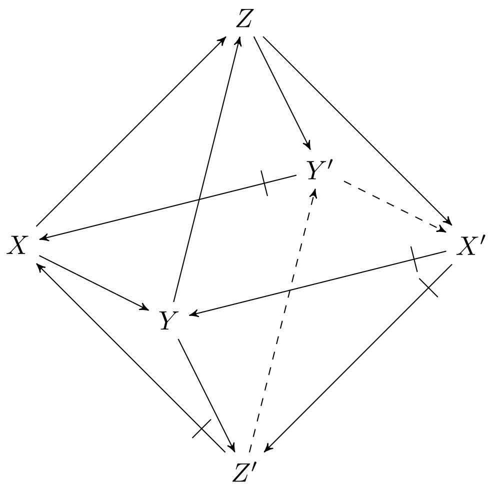
Remark 27.2. Sometimes morphisms $Z \longrightarrow X[1]$ are denoted by arrows

$$\begin{tikzcd} Z \arrow[r, "|" marking] & X. \end{tikzcd}$$

Then the elements of Δ can be depicted as actual triangles

$$\begin{tikzcd}
X \arrow[rr] & & Y \arrow[dl] \\
 & Z \arrow[ul, "|" marking] &
\end{tikzcd}$$

In particular the octahedron in Axiom (T4) becomes visible in this notation:



Here all the oriented triangles lie in Δ, and all the non-oriented triangles and squares commute.

Remark 27.3. (T3), by use of (T2), can be seen as a kind of "2 out of 3"-property: Given any two morphisms connecting two triangles, one may find a third.

Theorem 27.4 (Long exact Hom-sequence)**.** *Let $\mathscr{T}$ be a triangulated category, $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in Δ, and $T \in \mathfrak{Ob}\,\mathscr{T}$. Then the sequences*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T, X[n]) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T, Y[n]) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T, Z[n]) \longrightarrow$$
$$\mathrm{Hom}_{\mathscr{T}}(T, X[n+1]) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T, Y[n+1]) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T, Z[n+1]) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Z[n], T) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y[n], T) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(X[n], T) \longrightarrow$$
$$\mathrm{Hom}_{\mathscr{T}}(Z[n-1], T) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Y[n-1], T) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(X[n-1], T) \longrightarrow \cdots$$

are exact.

Proof. We prove the first claim, the second one is dual. (Note that $\mathscr{T}^{\mathrm{op}}$ is also triangulated.)

By the rotation axiom (T2) it suffices to check that the sequence

$$\mathrm{Hom}_{\mathscr{T}}(T,X) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T,Y) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(T,Z)$$

is exact. We do so by comparing the given triangle to the trivial triangle $T \to T \to 0 \to T[1]$.

$$\begin{tikzcd}
T \arrow[r] \arrow[d, dashed, "g"] & T \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & T[1] \arrow[d, dashed, "g[1]"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

By (T3) the existence of the dashed arrow g is equivalent to the existence of the middle dashed arrow. That is, for $f \in \mathrm{Hom}_{\mathscr{T}}(T,Y)$ we have

$$[Y \longrightarrow Z] \circ f = 0 \iff \exists g \in \mathrm{Hom}_{\mathscr{T}}(T,X)\colon [X \longrightarrow Y] \circ g = f. \qquad \square$$

Remark 27.5. The above theorem says that any morphism in a triangle is a weak kernel of the next morphism, and a weak cokernel of the previous morphism.

Theorem 27.6 (2 out of 3 property for isomorphisms)**.** *Let $\mathscr{T}$ be a triangulated category, and consider two triangles connected by morphisms as in the following diagram.*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f"] & Y_1 \arrow[r] \arrow[d, "g"] & Z_1 \arrow[r] \arrow[d, "h"] & X_1[1] \arrow[d, "f[1]"] \\
X_2 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] & X_2[1]
\end{tikzcd}$$

If two of the morphisms f, g, and h are isomorphisms, then so is the third one.

Proof. By (T2) we may rotate the triangles and assume f and g are isomorphisms. Now we apply the functor $\mathrm{Hom}_{\mathscr{T}}(-,Z_1)$ to the entire diagram, obtaining

$$\begin{tikzcd}
(X_1, Z_1) & (Y_1, Z_1) \arrow[l] & (Z_1, Z_1) \arrow[l] & (X_1[1], Z_1) \arrow[l] & (Y_1[1], Z_1) \arrow[l] \\
(X_2, Z_1) \arrow[u, "- \circ f"'] & (Y_2, Z_1) \arrow[l] \arrow[u, "- \circ g"'] & (Z_2, Z_1) \arrow[l] \arrow[u, "- \circ h"'] & (X_2[1], Z_1) \arrow[l] \arrow[u, "- \circ f[1]"'] & (Y_2[1], Z_1) \arrow[l] \arrow[u, "- \circ f[1]"']
\end{tikzcd}$$

where $(\star, Z_1)$ is short for $\mathrm{Hom}_{\mathscr{T}}(\star, Z_1)$.

Since f and g are isomorphisms it follows that also the left two and right two vertical maps in this diagram are isomorphisms. Now, by the five lemma, the morphism

$$- \circ h \colon \mathrm{Hom}_{\mathscr{T}}(Z_2, Z_1) \longrightarrow \mathrm{Hom}_{\mathscr{T}}(Z_1, Z_1)$$

is an isomorphism. In particular there is $\tilde{h} \in \mathrm{Hom}_{\mathscr{T}}(Z_2, Z_1)$ such that $\tilde{h} \circ h = \mathrm{id}_{Z_1}$, that is h is split mono.

Similarly, using the functor $\mathrm{Hom}_{\mathscr{T}}(Z_2, -)$, one sees that h is split epi. Thus h is an isomorphism. $\square$

28 Homotopy categories are triangulated

Theorem 28.1. *Assume $\mathscr{A}$ is an additive category. Then the homotopy category $\mathbf{K}(\mathscr{A})$ is triangulated, with Δ being the class of all diagrams isomorphic to standard triangles $A^\bullet \xrightarrow[f^\bullet]{} B^\bullet \longrightarrow \mathrm{Cone}(f^\bullet) \longrightarrow A^\bullet[1]$.*

Proof. We have seen (the first half of) (T2) in Proposition 26.1.

The first and last point of (T1) hold by construction, for the second one note that $0 \longrightarrow X^\bullet \xrightarrow{\mathrm{id}_X} X^\bullet \longrightarrow 0$ is a triangle, so, since we already checked (T2), so is $X^\bullet \xrightarrow{\mathrm{id}_X} X^\bullet \longrightarrow 0 \longrightarrow X^\bullet[1]$.

For (T3) we may, up to isomorphism, assume the following setup:

$$\begin{tikzcd}
X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "u^\bullet"] & Y^\bullet \arrow[r, "\binom{1}{0}"] \arrow[d, "v^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, dashed, "\exists ?"] & X^\bullet[1] \arrow[d, "u^\bullet[1]"] \\
X'^\bullet \arrow[r, "f'^\bullet"] & Y'^\bullet \arrow[r, "\binom{1}{0}"] & \mathrm{Cone}(f'^\bullet) \arrow[r, "(0\ 1)"] & X'^\bullet[1]
\end{tikzcd}$$

One easily sees that the map given by $w^n = \left(\begin{smallmatrix} v^n & 0 \\ 0 & u^{n+1} \end{smallmatrix}\right)$ fits into this diagram.

To check the octahedral axiom (T4), we again may assume that all triangles are standard triangles, that is consider the commutative diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "f^\bullet"] \arrow[d, equal] & Y^\bullet \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "g^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} g^n & 0 \\ 0 & 1 \end{smallmatrix}\right)_n"] & X^\bullet[1] \arrow[d, equal] \\
X^\bullet \arrow[r, "g^\bullet \circ f^\bullet"] & Z^\bullet \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \mathrm{Cone}(g^\bullet \circ f^\bullet) \arrow[r, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & X^\bullet[1] \\
 & \mathrm{Cone}(g^\bullet) \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)", "\star"'] \arrow[d, "(0\ 1)"] & \mathrm{Cone}(\left(\begin{smallmatrix} g^n & 0 \\ 0 & 1 \end{smallmatrix}\right)_n) \arrow[d, "\left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)"] & \\
 & Y^\bullet[1] \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \mathrm{Cone}(f^\bullet)[1] &
\end{tikzcd}$$

The map marked $\star$ in the diagram above is an isomorphism in $\mathbf{K}(\mathscr{A})$, with inverse given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f^{n+1} & 1 & 0 \end{pmatrix}_n : \mathrm{Cone}(\left(\begin{smallmatrix} g^n & 0 \\ 0 & 1 \end{smallmatrix}\right)_n) \longrightarrow \mathrm{Cone}(g^\bullet).$$

It only remains to check that the square

$$\begin{tikzcd}
\mathrm{Cone}(g^\bullet \circ f^\bullet) \arrow[rr, "(0\ 1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & & X^\bullet[1] \arrow[d, "f^\bullet[1]"] \\
\mathrm{Cone}(\left(\begin{smallmatrix} g^n & 0 \\ 0 & 1 \end{smallmatrix}\right)_n) \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & f^{n+1} & 1 & 0 \end{smallmatrix}\right)_n", "\cong"'] & \mathrm{Cone}(g^\bullet) \arrow[r, "(0\ 1)"] & Y^\bullet[1]
\end{tikzcd}$$

commutes up to homotopy. In fact one easily checks that it even commutes in $\mathbf{C}(\mathscr{A})$. $\square$

Observation 28.2. Any object $A \in \mathcal{O}b\,\mathscr{A}$ may be considered as a complex $\cdots 0 \longrightarrow X \longrightarrow 0 \longrightarrow \cdots$, with X in degree 0. This construction gives a fully faithful embedding of $\mathscr{A}$ into $\mathbf{C}(\mathscr{A})$ and into $\mathbf{K}(\mathscr{A})$. (Note that no non-zero map between complexes of this form can be null-homotopic.)

By abuse of notation, we identify the object X with the complex as above.

Lemma 28.3. *Let $X \in \mathscr{A}$, and $\mathscr{A}^\bullet \in \mathbf{C}(\mathscr{A})$. Then we may consider the complex*

$$\operatorname{Hom}_{\mathscr{A}}(X, A^\bullet).$$

We have

$$\begin{aligned} \mathtt{Z}^n \operatorname{Hom}_{\mathscr{A}}(X, A^\bullet) &= \operatorname{Hom}_{\mathbf{C}(\mathscr{A})}(X, A^\bullet[n]) \qquad \textit{and} \\ \mathtt{H}^n \operatorname{Hom}_{\mathscr{A}}(X, A^\bullet) &= \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(X, A^\bullet[n]). \end{aligned}$$

Dually

$$\begin{aligned} \mathtt{Z}^n \operatorname{Hom}_{\mathscr{A}}(A^\bullet, X) &= \operatorname{Hom}_{\mathbf{C}(\mathscr{A})}(A^\bullet, X[n]) \qquad \textit{and} \\ \mathtt{H}^n \operatorname{Hom}_{\mathscr{A}}(A^\bullet, X) &= \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A^\bullet, X[n]). \end{aligned}$$

Proof. We see that

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}(\mathscr{A})}(X, A^\bullet[n]) &= \{\varphi \in \operatorname{Hom}_{\mathscr{A}}(X, A^n) \mid d_A^n \circ \varphi = 0\} \\ &= \operatorname{Ker}[\operatorname{Hom}_{\mathscr{A}}(X, A^n) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(X, A^{n+1})] \\ &= \mathtt{Z}^n \operatorname{Hom}_{\mathscr{A}}(X, A^\bullet). \end{aligned}$$

Moreover a morphism from X to $A^\bullet[n]$ is null-homotopic if and only if if factors through d_A^{n-1}, that is lies in

$$\mathtt{B}^n \operatorname{Hom}_{\mathscr{A}}(X, A^\bullet) = \operatorname{Im}[\operatorname{Hom}_{\mathscr{A}}(X, A^{n-1}) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(X, A^n)].$$

The claim on homology now follows by taking quotients. □

Recall that, provided an abelian category $\mathscr{A}$ has enough projectives, we have the functor $\mathtt{p}\colon \mathscr{A} \longrightarrow \mathbf{K}(\mathscr{A})$ taking an object to its projective resolution. Recall also that, by the horseshoe lemma (Proposition 20.8), for a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad \text{in } \mathscr{A}$$

we have a triangle

$$\mathtt{p}A \longrightarrow \mathtt{p}B \longrightarrow \mathtt{p}C \longrightarrow \mathtt{p}A[1] \qquad \text{in } \mathbf{K}(\mathscr{A}).$$

Theorem 28.4. *Assume $\mathscr{A}$ has enough projectives. Then*

$$\operatorname{Ext}^n_{\mathscr{A}}(A, B) = \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(\mathtt{p}A, B[n]).$$

Dually, if $\mathscr{A}$ has enough injectives, then

$$\operatorname{Ext}^n_{\mathscr{A}}(A, B) = \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A, \mathtt{i}B[n]).$$

Proof. We have $\mathrm{Ext}^n_{\mathscr{A}}(A, B) = \mathtt{H}^n \operatorname{Hom}_{\mathscr{A}}(\mathtt{p}A, B)$ by definition. Now the claim follows from Lemma 28.3 above. $\square$

Remark 28.5. In view of this theorem, the long exact Hom-Ext-sequence can be seen as a long exact sequence coming from a triangle in the homotopy category.

We proceed by extending the above to arbitrary derived functors. To do so, we need the following two observations:

Observation 28.6. Let $\mathscr{A}$ be an abelian category. Taking homology $\mathtt{H}^0$ takes triangles in $\mathbf{K}(\mathscr{A})$ to long exact sequences. (This is just a restatement of the long exact sequence of homology – see Theorem 17.6.)

Observation 28.7. Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be any additive functor. Then $\mathtt{F}_{\mathbf{K}}$ preserves triangles.

Construction 28.8. Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be right exact. Then the long exact sequence of derived functors (associated to a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathscr{A}$) is the long exact sequence of homology coming from the triangle

$$\mathtt{F}_{\mathbf{K}}\mathtt{p}A \longrightarrow \mathtt{F}_{\mathbf{K}}\mathtt{p}B \longrightarrow \mathtt{F}_{\mathbf{K}}\mathtt{p}C \longrightarrow \mathtt{F}_{\mathbf{K}}\mathtt{p}A[1] \qquad \text{in } \mathbf{K}(\mathscr{B}).$$

29 Derived categories

Derived categories address the following two (closely related) issues with homotopy categories:

- Short exact sequences are not triangles in the homotopy category. (However one may get triangles replacing the objects by projective or injective resolutions.)

- Quasi-isomorphisms preserve all information on homology, but are not isomorphisms in the category $\mathbf{K}(\mathscr{A})$. In particular, in the discussion above, we had to take a projective or injective resolution, instead of the object itself (which is quasi-isomorphic).

The answer to these issues it to (brute force) make quasi-isomorphisms invertible.

Construction 29.1. Let $\mathscr{A}$ be an abelian category. A *roof* from a complex $X^\bullet$ to $Y^\bullet$ is a diagram of the form

$$\begin{tikzcd} & \tilde{X}^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

with some middle object $\tilde{X}^\bullet$, and where q is a quasi-isomorphism. For compact notation we write the above roof as $f \cdot q^{-1}$.

Two roofs $f \cdot q^{-1}$ and $g \cdot r^{-1}$ are called equivalent if there is a commutative diagram

$$\begin{tikzcd} & \tilde{X}^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & \\ X^\bullet & H^\bullet \arrow[u, "\text{qis}"] \arrow[d, "\text{qis}"'] & Y^\bullet \\ & \tilde{\tilde{X}}^\bullet \arrow[ul, "\text{qis } r"] \arrow[ur, "g"'] & \end{tikzcd}$$

Remark 29.2. In other words, if we denote the middle quasi-isomorphisms by q' and r' respectively, we find a common denominator $q \circ q' = r \circ r'$, and then compare the enumerators $f \circ q'$ and $g \circ r'$.

We need to check that the above notion of equivalence defines an equivalence relation. To that end (and in fact throughout the discussion of roofs) we need the following observation.

Lemma 29.3 (Ore condition)**.** *Let $\mathscr{A}$ be an abelian category. Given the solid part of the following square, where q is a quasi-isomorphism, it is possible to find the dashed part (including $\tilde{Y}^\bullet$), where r is a quasi isomorphism.*

$$\begin{tikzcd} \tilde{X}^\bullet \arrow[r, "f"] \arrow[d, "qis\ q"'] & Y^\bullet \arrow[d, dashed, "r\ qis"] \\ X^\bullet \arrow[r, dashed, "g"] & \tilde{Y}^\bullet \end{tikzcd}$$

Dually, given the dashed part, it is possible to find the solid part.

Proof. We complete q to a triangle as in the upper row of the following diagram

$$\begin{tikzcd}
\tilde{X}^\bullet \arrow[r, "q"] \arrow[d, "f"'] & X^\bullet \arrow[r] \arrow[d, dashed, "g"] & \mathrm{Cone}(q) \arrow[r] \arrow[d, "\mathrm{id}_{\mathrm{Cone}(q)}"] & \tilde{X}^\bullet[1] \arrow[d, "f[1]"] \\
Y^\bullet \arrow[r, dashed, "r"] & \tilde{Y}^\bullet \arrow[r, dashed] & \mathrm{Cone}(q) \arrow[r] & Y^\bullet[1]
\end{tikzcd}$$

Now by (T1) and (T3) we can complete the diagram as indicated by the dashed arrows, such that the lower row is also a triangle. Since q is quasi-iso we know that $\mathrm{Cone}(q)$ is exact. Now, since the cone of r is (isomorphic to) $\mathrm{Cone}(q)$, it follows that also r is a quasi-isomorphism. $\square$

Lemma 29.4. *The above defines an equivalence relation on the collection of roofs from $X^\bullet$ to $Y^\bullet$.*

Proof. The definition of equivalence is clearly reflexive and symmetric.

Assume $f \cdot q^{-1}$ is equivalent to $g \cdot r^{-1}$, which in turn is equivalent to $h \cdot s^{-1}$, as in the solid part of the following diagram.

$$\begin{tikzcd}
& \tilde{X}^\bullet \arrow[ddl, "\text{qis } q"'] \arrow[ddrr, "f"] & & \\
& H^\bullet \arrow[u, "\text{qis}"] \arrow[d, "\text{qis}"] & \hat{H}^\bullet \arrow[l, dashed, "\text{qis}"'] \arrow[ddl, dashed, "\text{qis}"] & \\
X^\bullet & \tilde{\tilde{X}}^\bullet \arrow[l, "r", "\text{qis}"'] \arrow[rr, "g"] & & Y^\bullet \\
& \tilde{H}^\bullet \arrow[u, "\text{qis}"] \arrow[d, "\text{qis}"] & & \\
& \tilde{\tilde{\tilde{X}}}^\bullet \arrow[uul, "\text{qis } s"] \arrow[uurr, "h"'] & &
\end{tikzcd}$$

By the Ore condition (Lemma 29.3) is is possible to find $\hat{H}^\bullet$ and the two dashed quasi-isomorphisms such that the square in the middle commutes. (One easily sees that if three sides of a square are quasi-iso, then so is the forth.)

Now the claim follows by considering $\hat{H}^\bullet$ between the two outer roofs. $\square$

Construction 29.5. Let $\mathscr{A}$ be an abelian category. Assume that for any complexes $X^\bullet$ and $Y^\bullet$, the collection of roofs from $X^\bullet$ to $Y^\bullet$ up to equivalence is a set. Then we define the *derived category* by

$$\begin{aligned} \mathcal{Ob}\, \mathbf{D}(\mathscr{A}) &= \mathcal{Ob}\, \mathbf{K}(\mathscr{A}) \qquad \text{and} \\ \operatorname{Hom}_{\mathbf{D}(\mathscr{A})}(X^\bullet, Y^\bullet) &= \{\text{roofs from } X^\bullet \text{ to } Y^\bullet\}/\sim, \end{aligned}$$

with composition given as follows:

Given $f \cdot q^{-1} \colon X^\bullet \longrightarrow Y^\bullet$, and $g \cdot r^{-1} \colon Y^\bullet \longrightarrow Z^\bullet$ as in the solid part of the following diagram,

$$\begin{tikzcd}
 & & \tilde{\tilde{X}}^\bullet \arrow[dl, dashed, "\text{qis } \tilde{r}"'] \arrow[dr, dashed, "\tilde{f}"] & & \\
 & \tilde{X}^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & & \tilde{X}^\bullet \arrow[dl, "\text{qis } r"'] \arrow[dr, "g"] & \\
X^\bullet & & Y^\bullet & & Y^\bullet
\end{tikzcd}$$

we may find $\tilde{\tilde{X}}^\bullet$ and the two dashed maps by the Ore condition (Lemma 29.3) . We now define the product to be

$$(g \cdot r^{-1}) \circ (f \cdot q^{-1}) = (g\tilde{f}) \cdot (q\tilde{r})^{-1}.$$

One may check that this is well-defined up to equivalence of roofs, and only depends on the equivalence class of the factors. Then it is easy to see that this multiplication is associative.

Observation 29.6.

- The derived category comes with a natural functor $\mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$ which is sends every complex to itself, and a morphism f to the trivial roof $f \cdot \mathrm{id}^{-1}$.
- A complex $X^\bullet$ becomes isomorphic to 0 in $\mathbf{D}(\mathscr{A})$ if and only if it is exact.

- A morphism f in $\mathbf{K}(\mathscr{A})$ is mapped to the zero-morphism in $\mathbf{D}(\mathscr{A})$ if there is a quasi-isomorphism q such that $f \circ q = 0$. One can prove that this is equivalent to f factoring through an exact complex. (To see this, consider the cone of q.)
- For a quasi-isomorphism q, also the shift $q[n]$ is a quasi-isomorphism for any n. It follows that $[1]$ defines an autoequivalence of $\mathbf{D}(\mathscr{A})$.

Theorem 29.7. *Let $\mathscr{A}$ be an abelian category, such that $\mathbf{D}(\mathscr{A})$ is defined. Then $\mathbf{D}(\mathscr{A})$ is a triangulated category, where Δ consists of all triangles isomorphic to standard triangles*

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow X^\bullet[1],$$

where f is a morphism of complexes.

Proof. We check the axioms. Here we make heavy use of the fact that we already checked the axioms for $\mathbf{K}(\mathscr{A})$.

(T1) For the first bullet point (there is a triangle starting with any morphism) we proceed as follows: Given the morphism $f \cdot q^{-1}$ we first find a standard triangle starting with f, and then alter it by the isomorphism q.

The second bullet point (triangle with identity as first morphism) follows from the same statement for $\mathbf{K}(\mathscr{A})$.

The third one (Δ closed under isos) holds by definition.

(T2) Up to isomorphism, the triangle is a standard triangle. For such triangles we know that the rotations are isomorphic to standard triangles in $\mathbf{K}(\mathscr{A})$, and therefore also in $\mathbf{D}(\mathscr{A})$.

(T3) Up to isomorphism we may assume that the two triangles we want to connect are standard triangles. That is we have the solid part of the following diagram.

$$\begin{tikzcd}
 & & A^\bullet \arrow[rr, "a"] & & B^\bullet \arrow[rr] & & \mathrm{Cone}(a) \arrow[rrr] & & & A^\bullet[1] \\
 & \tilde{A} \arrow[ur, "\text{qis } q"] \arrow[ddr, "f"'] & & & & & & & \tilde{A}[1] \arrow[ur, "\text{qis } q[1]"] \arrow[ddr, "f[1]"] & \\
\tilde{\tilde{A}} \arrow[ur, dashed, "\text{qis } \tilde{q}"] \arrow[rrr, dashed, "\tilde{a}"] & & & \tilde{B} \arrow[uur, "\text{qis } r"] \arrow[dr, "g"] & & \mathrm{Cone}(\tilde{a}) \arrow[uur, dotted, "s"] \arrow[dr, dotted, "h"] \arrow[rr, dotted] & & \tilde{\tilde{A}}[1] \arrow[ur, dashed, "\text{qis } \tilde{q}[1]"] & & \\
 & & C^\bullet \arrow[rr, "a"] & & D^\bullet \arrow[rr] & & \mathrm{Cone}(c) \arrow[rrr] & & & C^\bullet[1]
\end{tikzcd}$$

By the Ore condition (Lemma 29.3) we can find the two dashed maps as in the diagram above, such that the upper pentagon commutes. We may even choose them in such a way that also the lower pentagon commutes.

We complete $\tilde{a}$ to a triangle, and apply (T3) for the homotopy category to find the morphisms s and h making the diagram commutative. Taking homology and applying the five lemma (Theorem 13.1), we can see that r and $q \circ \tilde{q}$ being quasi-isomorphisms automatically also makes s a quasi-isomorphism. Thus the morphism $h \cdot s^{-1}$ in $\mathbf{D}(\mathscr{A})$ is the desired morphism between cones.

(T4) For the octahedral axiom, one may argue (similarly to the above) that all the input data lies in the homotopy category, and thus (T4) follows from the same axiom for $\mathbf{K}(\mathscr{A})$. $\square$

Remark 29.8. It might seem like we gained little, since the triangles in the derived category are "the same" as the triangles in the homotopy category. However, since there are now more isomorphisms (all quasi-isos have become isomorphisms), there are in fact "more" triangles.

Example 29.9. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in an abelian category. Then there is a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1] \text{ in } \mathbf{D}(\mathscr{A}).$$

To see this, consider the standard triangle

$$A \xrightarrow{f} B \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \operatorname{Cone}(f) \xrightarrow{(0\ 1)} A[1].$$

We note that $\operatorname{Cone}(f)$ is the complex in the upper row of the following diagram, and that the vertical map here is a quasi-isomorphism

$$\begin{tikzcd} {} \arrow[d, "q"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"'] & 0 \arrow[r] \arrow[d] & \cdots \\ {} & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Thus we also have the (isomorphic) triangle

$$A \xrightarrow{f} B \xrightarrow{q \circ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} C \xrightarrow{(0\ 1) \cdot q^{-1}} A[1]$$

in $\mathbf{D}(\mathscr{A})$. Finally note that $q \circ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = g$.

Theorem 29.10. *Let $\mathscr{A}$ be abelian, and $A, B \in \mathfrak{Ob}\,\mathscr{A}$. Then*

$$\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, B[n]) = \begin{cases} 0 & \text{if } n < 0, \\ \mathrm{Hom}_{\mathscr{A}}(A, B) & \text{if } n = 0, \\ \mathrm{YExt}^1_{\mathscr{A}}(A, B) & \text{if } n = 1. \end{cases}$$

Proof. We consider roofs

$$\begin{tikzcd} & E^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & \\ A & & B[n] \end{tikzcd}$$

where q is a quasi-isomorphism.

We first consider the truncation of $E^\bullet$ to the right as in the following diagram.

$$\begin{tikzcd} \tau^{\leqslant 0}E^\bullet : & \cdots \arrow[r] & E^{-1} \arrow[r] \arrow[d] & \operatorname{Ker} d_E^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ E^\bullet : & \cdots \arrow[r] & E^{-1} \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & \cdots \end{tikzcd}$$

One may observe that the natural map $r\colon \tau^{\leqslant 0}E^\bullet \longrightarrow E^\bullet$ indicated above induces an isomorphism on all non-positive homologies. Since in our setup $E^\bullet$ is quasi-isomorphic to A it has non-zero homology only in degree 0. Therefore r is a quasi-isomorphism.

It follows that the roof $f \cdot q^{-1}$ is equivalent to the roof $fr \cdot (qr)^{-1}$. In other words, up to equivalence we may assume that $E^\bullet$ is concentrated in non-positive degrees.

This proves the first claim, since $B[n]$ is concentrated in (the in that case positive) degree $-n$.

Now assume $n \geqslant 0$. Then we may (similarly to the above) cut off the left part of $E^\bullet$ as indicated in the following diagram.

$$\begin{tikzcd} E^\bullet : & \cdots \arrow[r] & E^{-n-1} \arrow[r] \arrow[d] & E^{-n} \arrow[r] \arrow[d] & E^{-n+1} \arrow[r] \arrow[d] & \cdots \\ \tau^{\geqslant -n}E^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & \operatorname{Cok} d_E^{-n-1} \arrow[r] & E^{-n+1} \arrow[r] & \cdots \end{tikzcd}$$

As before we see that the map $s\colon E^\bullet \longrightarrow \tau^{\geqslant -n}E^\bullet$ is a quasi-isomorphism (since $n \geqslant 0$).

Now observe that both q and f factor through s (since both A and $B[n]$ are concentrated in degrees $\geqslant -n$), say via q' and f'. Then we see that the roof $f \cdot q^{-1}$ is equivalent to $f' \cdot (q')^{-1}$. Thus now we may assume that $E^\bullet$ is concentrated in degrees $-n, \ldots, 0$.

Now we consider the case $n = 0$. Then, by the above discussion, we may assume that $E^\bullet$ is concentrated in degree 0. Thus q is an isomorphism, and hence $f \cdot q^{-1}$ lies in the image of the natural map $\mathrm{Hom}_{\mathscr{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, B)$. Conversely this map is also injective, since no non-zero morphism from A to B has vanishing homology. ☐

Proposition 29.11. *Let $\mathscr{A}$ be an abelian category, and $P^\bullet$ a right bounded complex of projectives. (That is all P^n are projective, and $\exists N \forall n > N\colon P^n = 0$.) Then*

(1) Let $E^\bullet$ be an exact complex. Then $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, E^\bullet) = 0$.

(2) Any quasi-isomorphism $\tilde{P}^\bullet \longrightarrow P^\bullet$ from any complex $\tilde{P}^\bullet$ to $P^\bullet$ is a split epimorphism in the category $\mathbf{K}(\mathscr{A})$.

(3) Let $X^\bullet$ be any complex. Then the map

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, X^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(P^\bullet, X^\bullet)$$

is an isomorphism.

Proof. (1) Let $f^\bullet\colon P^\bullet \longrightarrow E^\bullet$ be a morphism of complexes. We construct a null-homotopy iteratedly from right to left. So let n be some index, and assume we already have $h^i\colon P^i \longrightarrow E^{i-1}$ for $i > n$, such that $f^i = d_E^{i-1} \circ h^i + h^{i+1} d_P^i$. (Note that this is automatic for $n \geqslant N$ - thus we have a starting point for our iterated construction.)

The setup is depicted in the following diagram.

$$
\begin{tikzcd}
 & & P^n \arrow[r, "d_P^n"] \arrow[d, "f^n"] \arrow[dll, dotted, "h^n"'] \arrow[ddl, dashed] & P^{n+1} \arrow[d, "f^{n+1}"] \arrow[dl, "h^{n+1}"'] \\
E^{n-1} \arrow[rr, "d_E^{n-1}"] \arrow[dr] & & E^n \arrow[r, "d_E^n"'] & E^{n+1} \\
 & \operatorname{Im} d_E^{n-1} \arrow[ur, tail] & &
\end{tikzcd}
$$

We observe that $d_E^n \circ (f^n - h^{n+1} \circ d_P^n) = 0$, and hence $f^n - h^{n+1} \circ d_P^n$ factors through $\operatorname{Im} d_E^{n-1} = \operatorname{Ker} d_E^n$ as indicated by the dashed arrow above. Since P^n is projective we may lift along the epimorphism $E^{n-1} \twoheadrightarrow \operatorname{Im} d_E^{n-1}$, obtaining h^n as indicated by the dotted arrow.

(2) Let $q\colon \tilde{P}^\bullet \longrightarrow P^\bullet$ be a quasi-isomorphism. Then, in the triangle

$$\tilde{P}^\bullet \xrightarrow{q} P^\bullet \longrightarrow \operatorname{Cone}(q) \longrightarrow \tilde{P}^\bullet[1]$$

in $\mathbf{K}(\mathscr{A})$, the complex $\operatorname{Cone}(q)$ is exact (Corollary 18.6), so by (1) the middle map vanishes.

It now follows that q is a split epimorphism.

(3) By (2) in any fraction $f \cdot q^{-1}\colon P^\bullet \longrightarrow X^\bullet$ the quasi-isomorphism q is a split epimorphism. Thus we may find $\tilde{q}$ such that $q \circ \tilde{q} = \mathrm{id}_{P^\bullet}$. One easily checks that $f \cdot q^{-1} = (f\tilde{q}) \cdot \mathrm{id}^{-1}$, so it lies in the image of the functor $\mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$.

On the other hand, we know that a morphism f in $\mathbf{K}(\mathscr{A})$ vanishes in $\mathbf{D}(\mathscr{A})$ if and only if there is a quasi-isomorphism q such that $f \circ q = 0$ (Observation 29.6. However, by (2) such a quasi-isomorphism is a split epimorphism, hence $f = 0$. □

Corollary 29.12. *Assume $\mathscr{A}$ has enough projectives or enough injectives. Then*

$$\operatorname{Hom}_{\mathbf{D}(\mathscr{A})}(A, B[n]) = \operatorname{Ext}^n_{\mathscr{A}}(A, B)$$

for any n and objects A and B of $\mathscr{A}$.

Proof. Assume $\mathscr{A}$ has enough projectives, and let $\mathtt{p}A$ be a projective resolution of A. Then the natural projection $q\colon \mathtt{p}A \longrightarrow A$ is a quasi-isomorphism, and so

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(A, B[n]) &= \mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(\mathtt{p}A, B[n]) \\
&= \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(\mathtt{p}A, B[n]) && \text{(by Proposition 29.11)} \\
&= \mathrm{Ext}^n_{\mathscr{A}}(A, B) && \text{(by Theorem 28.4)}
\end{aligned}$$

□

Remark 29.13.
- By Example 29.9 a short exact sequence in $\mathscr{A}$ "is" a triangle in $\mathbf{D}(\mathscr{A})$. Thus, by Corollary 29.12 above, the long exact Hom-Ext sequence can be interpreted as the long exact Hom-sequence coming from this triangle.
- Corollary 29.12 and Theorem 29.10 also show that all definitions of Ext coincide, when they are defined (Yoneda-Ext, deriving by first argument, and deriving by second argument).

30 Derived functors

Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor. We observed that applying this functor position by position gives rise to a functor $\mathtt{F}_{\mathbf{K}}\colon \mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{K}(\mathscr{B})$. We would now like to do the same thing for derived categories, that is we would like to have a functor $\mathtt{F}_{\mathbf{D}}$ making the following square commutative

$$\begin{tikzcd}
\mathbf{K}(\mathscr{A}) \arrow[r, "\mathtt{F}_{\mathbf{K}}"] \arrow[d, "\pi_{\mathscr{A}}"'] & \mathbf{K}(\mathscr{B}) \arrow[d, "\pi_{\mathscr{B}}"] \\
\mathbf{D}(\mathscr{A}) \arrow[r, "\mathtt{F}_{\mathbf{D}}"] & \mathbf{D}(\mathscr{B})
\end{tikzcd}$$

where $\pi_{\mathscr{A}}$ and $\pi_{\mathscr{B}}$ are the canonical functors from the homotopy categories to the corresponding derived categories.

Unfortunately, however, it is not possible to find such a functor $\mathtt{F}_{\mathbf{D}}$ in general:

Lemma 30.1. *Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor between abelian categories. Then a functor $\mathtt{F}_{\mathbf{D}}\colon \mathbf{D}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{B})$ making the diagram above commutative exists if and only if $\mathtt{F}$ is exact.*

Proof. If the functor $\mathtt{F}$ is exact, then it preserves homology, and thus in particular $\mathtt{F}_{\mathbf{K}}$ preserves quasi-isomorphisms. It follows that we can define $\mathtt{F}_{\mathbf{D}}(f \cdot q^{-1}) = \mathtt{F}_{\mathbf{K}}(f) \cdot \mathtt{F}_{\mathbf{K}}(q)^{-1}$. (Note that $\mathtt{F}_{\mathbf{K}}$ preserves equivalence of roofs, so this is in fact well-defined.)

On the other hand, if $\mathtt{F}$ is not an exact functor, then there will be a short exact sequence $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ in $\mathscr{A}$ such that the image

$$0 \longrightarrow \mathtt{F}A_1 \longrightarrow \mathtt{F}A_2 \longrightarrow \mathtt{F}A_3 \longrightarrow 0$$

is not exact. Interpreting this sequence as an element of $\mathbf{K}(\mathscr{A})$, we see that the object is sent to 0 by $\pi_{\mathscr{A}}$, but not by $\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}$. Clearly this makes it impossible to obtain a commutative square as above. $\square$

Since it usually is not possible to find a functor $\mathtt{F}_{\mathbf{D}}$ as above, one is lead to consider functors that make the square "as commutative as possible".

Definition 30.2. Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor between abelian categories. A *(total) left derived functor* of $\mathtt{F}$ is a functor $\mathbf{L}\mathtt{F}\colon \mathbf{D}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{B})$, together with a natural transformation $\phi\colon \mathbf{L}\mathtt{F} \circ \pi_{\mathscr{A}} \longrightarrow \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}$, which is universal in the following sense:

For any other functor $\mathtt{G}\colon \mathbf{D}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{B})$, together with a natural transformation $\psi\colon \mathtt{G} \circ \pi_{\mathscr{A}} \longrightarrow \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}$, there is a unique natural transformation $\zeta\colon \mathtt{G} \longrightarrow \mathbf{L}\mathtt{F}$ such that $\psi = \phi \circ \zeta_{\pi_{\mathscr{A}}}$.

Dually, a *(total) right derived functor* of $\mathtt{F}$ is a functor $\mathbf{R}\mathtt{F}\colon \mathbf{D}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{B})$, together with a natural transformation $\phi\colon \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}} \longrightarrow \mathbf{R}\mathtt{F} \circ \pi_{\mathscr{A}}$ satisfying a dual universal property.

Remark 30.3. In general, there is no reason for a total left or right derived functor to exist.

However, if one does exist, then the universal property guarantees that it is unique (up to unique natural isomorphism). Therefore we can talk about *the* left derived functor or *the* right derived functor in this case.

It is a bit technical to construct total derived functors between the entire derived categories in general (and requires additional assumptions on $\mathscr{A}$). To simplify our situation here a bit we consider the full subcategory of right bounded complexes

$$\mathbf{C}^{-}(\mathscr{A}) = \{A^{\bullet} \mid \exists N\ \forall n > N\ A^n = 0\} \subset \mathbf{C}(\mathscr{A}),$$

and its counterparts $\mathbf{K}^-(\mathscr{A}) \subset \mathbf{K}(\mathscr{A})$ and $\mathbf{D}^-(\mathscr{A}) \subset \mathbf{D}(\mathscr{A})$. Similarly we may consider the category of left bounded complexes $\mathbf{C}^+(\mathscr{A})$, the homotopy category of left bounded complexes $\mathbf{K}^+(\mathscr{A})$, and the derived category of left bounded complexes $\mathbf{D}^+(\mathscr{A})$.

Proposition 30.4. *Assume $\mathscr{A}$ has enough projectives. For any right bounded complex $A^\bullet$ there is a right bounded complex $\mathtt{p}(A^\bullet)$ of projectives and a quasi-isomorphism $\mathtt{p}(A^\bullet) \longrightarrow A^\bullet$.*

This construction gives a functor

$$\mathtt{p}\colon \mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{K}^-(\mathscr{A})$$

which is left adjoint to projection $\pi\colon \mathbf{K}^-(\mathscr{A}) \longrightarrow \mathbf{D}^-(\mathscr{A})$. Moreover, the unit of the adjunction $\epsilon\colon \mathrm{id}_{\mathbf{D}^-(\mathscr{A})} \longrightarrow \pi\mathtt{p}$ is a natural isomorphism.

Proof. We construct $\mathtt{p}(A^\bullet)$ iteratedly from right to left. Assume all A^i with $i > n$ are already projective. Pick an epimorphism $P^n \longrightarrow A^n$ with P^n projective, and consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-2} \arrow[r] \arrow[d, "\mathrm{id}"] & A^{n-1} \prod_{A^n} P^n \arrow[r] \arrow[d] & P^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & A^{n-2} \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

where the map $P^n \longrightarrow A^{n+1}$ is composition, and the map $A^{n-2} \longrightarrow A^{n-1} \prod_{A^n} P^n$ is obtained from the pullback property. Since the pullback is taken along an epimorphism the middle square is in fact exact, and thus this morphism of complexes is a quasi-isomorphism.

Iterating this construction one obtains the desired quasi-isomorphism

$$\eta_{A^\bullet}\colon \mathtt{p}(A^\bullet) \longrightarrow A^\bullet.$$

Now we can first turn $\mathtt{p}$ into a functor $\mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{D}^-(\mathscr{A})$ by setting $\mathtt{p}(f) = \eta_B^{-1} \circ f \circ \eta_A$ for any morphism $f\colon A^\bullet \longrightarrow B^\bullet$. Since by Proposition 29.11(3) the morphism sets in the derived and homotopy category coincide on right bounded complexes of projectives, $\mathtt{p}$ defines a functor $\mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{K}^-(\mathscr{A})$.

The fact that $\mathtt{p}$ is left adjoint to π follows from

$$\mathrm{Hom}_{\mathbf{K}^-(\mathscr{A})}(\mathtt{p}A^\bullet, B^\bullet) \overset{29.11(3)}{\cong} \mathrm{Hom}_{\mathbf{D}^-(\mathscr{A})}(\mathtt{p}A^\bullet, B^\bullet) \cong \mathrm{Hom}_{\mathbf{D}^-(\mathscr{A})}(A^\bullet, B^\bullet),$$

where the second isomorphism is due to the fact that the quasi-isomorphism $\eta_{A^\bullet}$ becomes an isomorphism in the derived category.

Finally we note that the unit is given by $\epsilon_{A^\bullet} = (\eta_{A^\bullet})^{-1}$ – which is defined on the derived level. ☐

Now we can prove that total right derived functors can be understood using projective resolutions at least in the setup of right bounded complexes.

Theorem 30.5. *Let $\mathtt{F}\colon \mathscr{A} \longrightarrow \mathscr{B}$ be an additive functor between abelian categories.*

- *Assume that $\mathscr{A}$ has enough projectives. Then on the subcategories of right bounded complexes there is a total left derived functor*
$$\mathbf{L}\mathtt{F}\colon \mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{D}^-(\mathscr{B})$$
given by $\mathbf{L}\mathtt{F} = \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{p}$.
- *Dually, if $\mathscr{A}$ has enough injectives, then there is a total right derived functor*
$$\mathbf{R}\mathtt{F}\colon \mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{D}^-(\mathscr{B})$$
with respect to left bounded complexes, given by $\mathbf{R}\mathtt{F} = \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{i}$.

Proof. We only prove the first claim, the second one is dual.

First note that in the diagram

$$\begin{tikzcd}
\mathbf{K}^-(\mathscr{A}) \arrow[r, "\mathtt{F}_{\mathbf{K}}"] \arrow[d, "\pi_{\mathscr{A}}"'] & \mathbf{K}^-(\mathscr{B}) \arrow[d, "\pi_{\mathscr{B}}"] \\
\mathbf{D}^-(\mathscr{A}) \arrow[r, "\mathbf{L}\mathtt{F}"] & \mathbf{D}^-(\mathscr{B})
\end{tikzcd}$$

we do have a natural transformation $\phi\colon \mathbf{L}\mathtt{F} \circ \pi_{\mathscr{A}} \longrightarrow \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}$. Recalling that $\mathtt{p}$ is left adjoint to $\pi_{\mathscr{A}}$ (see Proposition 30.4 above) we have the counit $\eta\colon \mathtt{p} \circ \pi_{\mathscr{A}} \longrightarrow \mathrm{id}_{\mathbf{K}^-(\mathscr{A})}$. We now choose

$$\phi = (\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}})(\eta)\colon \underbrace{\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{p}}_{=\mathbf{L}\mathtt{F}} \circ \pi_{\mathscr{A}} \longrightarrow \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}.$$

It only remains to verify that our choice of $\mathbf{L}\mathtt{F}$ and ϕ satisfy the universal property of Definition 30.2. Let $\mathtt{G}\colon \mathbf{D}^-(\mathscr{A}) \longrightarrow \mathbf{D}^-(\mathscr{B})$ be a different functor, together with a natural transformation $\psi\colon \mathtt{G} \circ \pi_{\mathscr{A}} \longrightarrow \pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}}$. If $\zeta\colon \mathtt{G} \longrightarrow \mathbf{L}\mathtt{F}$ is a natural transformation such that $\psi = \phi \circ \zeta_{\pi_{\mathscr{A}}}$ then

$$\psi_{\mathtt{p}} = \phi_{\mathtt{p}} \circ \zeta_{\pi_{\mathscr{A}} \circ \mathtt{p}}$$

Since the unit $\epsilon\colon \mathrm{id}_{\mathbf{D}^-(\mathscr{A})} \longrightarrow \pi_{\mathscr{A}} \circ \mathtt{p}$ is a natural isomorphism, and since

$$\phi_{\mathtt{p}} = (\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}})(\eta_{\mathtt{p}}) = (\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}} \circ \mathtt{p})(\epsilon^{-1})$$

we obtain

$$\zeta = \mathbf{L}\mathtt{F}(\epsilon)^{-1} \circ \zeta_{\pi_{\mathscr{A}} \circ \mathtt{p}} \circ \mathtt{G}(\epsilon) = \psi_{\mathtt{p}} \circ \mathtt{G}(\epsilon).$$

In particular ζ as uniquely determined.

Conversely, with the choice $\zeta = \psi_{\mathtt{p}} \circ \mathtt{G}(\epsilon)$, we obtain

$$\begin{aligned} \phi \circ \zeta_{\pi_{\mathscr{A}}} &= (\pi_{\mathscr{B}} \circ \mathtt{F}_{\mathbf{K}})(\eta) \circ \psi_{\mathtt{p}\pi_{\mathscr{A}}} \circ \mathtt{G}(\epsilon_{\pi_{\mathscr{A}}}) \\ &= \psi \circ (\mathtt{G} \circ \pi_{\mathscr{A}})(\eta) \circ \mathtt{G}(\epsilon_{\pi_{\mathscr{A}}}) \\ &= \psi \circ \mathtt{G}(\underbrace{\pi_{\mathscr{A}}(\eta) \circ \epsilon_{\pi_{\mathscr{A}}}}_{=\mathrm{id}_{\mathbf{D}^-(\mathscr{A})}}) \\ &= \psi. \end{aligned}$$

Thus our choice of $\mathbf{L}\mathtt{F}$ and ϕ does satisfy the universal property, so it is the total left derived functor of $\mathtt{F}$. □

Remark 30.6.
- Theorem 30.5 shows, in particular, that the appearance of projective and injective resolutions in the definition of left and right derived functors is not an arbitrary choice / coincidence. On the contrary, the definition of derived functors via a universal property forces this construction.
- In many cases an obvious analog of Theorem 30.5 also holds for unbounded complexes. However, for such a result one typically needs that $\mathscr{A}$ has certain colimits (essentially along the poset $\mathbb{Z}$), and that these colimits are exact.

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