Sobolevrom 2. 3. 2017

17. W. conv. and comp. in L^1

(X, μ) meas. sp.

Bnd'ness $\not\Rightarrow$ w. seq'al comp. in $L^1(X)$

Need also equiint.!

$\{f_n\}_n \subset L^1(X)$ is equiintegrable if:

(i) $\forall \varepsilon > 0 \ \exists B \subset X$ meas., $\mu(B) < \infty$ s.t.

$$\sup_n \int_{X \setminus B} |f_n| \, d\mu < \varepsilon \quad \text{(tightness!)}$$

(ii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$A \subset X, \ \mu(A) < \delta \Rightarrow \sup_n \int_A |f_n| \, d\mu < \varepsilon$$

Thm. 49 (Dunford-Pettis): $\{f_n\}_n \subset L^1(X)$.

$\{f_n\}_n$ w. seq'ally comp. in L^1

$$\Updownarrow$$

$\{f_n\}_n$ bnd. ($\sup_n \|f_n\|_1 < \infty$) and equiint.

Pf $\Uparrow$):

0) We may assume $f_n \geq 0$ [$f_n = f_n^+ - f_n^-$ and $\{f_n^+\}_n$ and $\{f_n^-\}_n$ also bnd. + equiint. Seq'al comp. of $\{f_n^+\}_n, \{f_n^-\}_n \Rightarrow$ seq'al comp. of $\{f_n\}_n$]

1) Must find subseq'nce $\{f_{n_k}\}_k$ and $f \in L^1$ s.t. (Riesz repr. thm.):

$$\int f_{n_k} g \to \int f g \quad \forall\, g \in L^\infty$$

Since simple func'ns dense in L^∞, it suffices to take $g = \chi_A$:

$$\int_A f_{n_k} \to \int_A f \quad \forall\, A \subset X \text{ meas.}$$

2) By (i) we may assume $\mu(X) < \infty$ [$X = B$ from (i)]

[Let, $A \subset X$, $B = B_m$ in (i) w. $\varepsilon = \frac{1}{m}$ and $\int_{A\cap B_m} f_{n_k} \to \int_{A \cap B_m} f$:

$$\Rightarrow \left|\int_A (f_{n_k} - f)\right| \le \underbrace{\left|\int_{A\cap B_m} (\quad)\right|}_{\substack{\xrightarrow[k\to\infty]{} 0 \\ \text{by assumption}}} + \underbrace{\left(\int_{X\setminus B_m} |f_{n_k}| + \int_{X \setminus B_m} |f|\right)}_{\substack{< 2\cdot\frac{1}{m} \\ \text{by (i)}}}$$

Take diag. seq'nce conv. on each $B_m \Rightarrow$ conv. in X]

3) Conv. subseq'nce f_{n_k} in $\mathcal{M}(X)$:

By bnd'ness and $\mu(X) < \infty$

$$\sup_n \sup_{\substack{\varphi \in C_0(X) \\ \|\varphi\|_\infty \le 1}} \left|\int f_n \varphi\right| \overset{\text{Hölder}}{\le} \sup_n \int |f_n| < \infty$$

Hence $\{f_n\, d\mu\}$ unif. bnd. in $\mathcal{M}(X) = (C_0)'$;

By seq'al comp. in $\mathcal{M}(X)$ (later?), $\exists\, f_{n_k}$ and $\nu \in \mathcal{M}(X)$ s.t.

$$\int f_{n_k} \varphi\, d\mu \to \int \varphi\, d\nu \quad \forall\, \varphi \in C_0(X)$$

4) f_{n_k} w. Cauchy in $L^1(X)$: (Fix $A \subset X$ and $\varepsilon > 0$, and take $\delta > 0$ from (ii).

Take $C_0 \ni \varphi_k \to \chi_A$ a.e. and $\|\varphi_k\|_\infty \le 1$

[e.g. $\varphi_k = \chi_A * \rho_k$]

By Egorov's thm. $\exists\, A_\delta \subset X$ meas. s.t.

$$\mu(X \setminus A_\delta) < \delta \quad \text{and} \quad \|\varphi_k - \chi_A\|_{L^\infty(A_\delta)} \to 0$$

Hence

$$\left|\int_{A_\delta}(f_{n_j}-f_{n_l}) \underset{\checkmark}{\overset{\chi_A}{}}\right|$$

$$\le \underbrace{\left|\int_{A_\delta}(f_{n_j}-f_{n_l})(\chi_A-\varphi_k)\right|}_{\le 2\underbrace{\sup\|f_{n_j}\|_1}_{=:M<\infty}\underbrace{\|\chi_A-\varphi_k\|_{L^\infty(A_\delta)}}_{<\frac{\varepsilon}{2M}\text{ if } k \text{ big (indep. of } j,l)}} + \underbrace{\left|\int_{A_\delta}(f_{n_j}-f_{n_l})\varphi_k\right|}_{<\varepsilon \text{ if } j,l \text{ big (dep. on } k)}$$

$< 2\varepsilon$ if k, j, l big enough

and by equiint (ii):

$$\left|\int_A (f_{n_j}-f_{n_l})\right|$$

$$\le \underbrace{\left|\int_{A\cap A_\delta}(\quad)\right|}_{<2\varepsilon} + \underbrace{\left|\int_{A\setminus A_\delta}(\quad)\right|}_{\le 2\sup_k \int_{A\setminus A_\delta}|f_{n_k}| \overset{(ii)}{<} 2\varepsilon} < 4\varepsilon$$

5) ~~Def $F(\varphi) = \lim_k \int_X f_{n_k}\varphi \quad \forall\, \varphi \in L^\infty$~~ Se p. 3,5

~~Well-def. by 4) and density of simple f'ns in L^∞: $F\in(L^\infty)'$~~

15:06 [$F(\varphi) = \int \varphi\, d\nu \quad \forall\, \varphi \in C_0$!]

False !! ~~By Riesz repr. thm., $\exists\, f \in L^1(X)$ s.t.~~ does not hold in L^∞

~~$$F(\varphi) = \int f\varphi \quad \forall\, \varphi\in L^\infty$$~~

~~and hence~~

~~$$\int f_{n_k}\varphi \to \int f\varphi \quad \forall\, \varphi \in L^\infty$$~~ $\square$

5) Def.: $\nu_k(A) = \int_A f_{n_k}\, d\mu$ $\quad \forall_A A \subset X$ meas.

$\tilde{\nu}(A) = \lim_k \nu_k(A)$

$\tilde{\nu}$ well-def. by 4); $\overset{\text{Chk:}}{\nu_k}$, $\tilde{\nu}$ pos. meas. on X. $(f_n \geq 0)$.

By equiint. of f_n; $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$A \subset X \text{ meas.}, \mu(A) < \delta \Rightarrow \sup_n \int_A f_n\, d\mu = \sup_n \int_A |f_n|\, d\mu < \varepsilon$$

(?)

$$\Rightarrow |\tilde{\nu}(A)| = \lim_k \int_A |f_{n_k}|\, d\mu < \varepsilon$$

$$\Rightarrow \tilde{\nu} \text{ abs. cont. w.r.t. } \mu \quad (\mu(B) = 0 \Rightarrow \tilde{\nu}(B) = 0)$$

By Radon–Nikodym $\exists f \in L^1(X,\mu)$ s.t.

$$\tilde{\nu} = f\mu,$$

and then

$$\int_A f_{n_k}\, d\mu = \nu_k(A) \to \tilde{\nu}(A) = \int_A f\mu \quad \forall A \subset X \text{ meas.}$$

$\square$

(Ambrosio – Fusco p. 14)

Fonseca – Leoni p. 175

Giaquinta – Modica – Soucek p. 55

18. Equiint. Also called uniform int.

Ex. 50: X bnd., f_n meas., $|f_n| \overset{a.e.}{\le} g \in L^1 \Rightarrow \{f_n\}_n$ equiint.:

$$\sup_n \int_A |f_n| \le \int_A g \le \int_A \tilde{g} + \frac{\varepsilon}{2} \le \|\tilde{g}\|_\infty \mu(A) + \frac{\varepsilon}{2} < \varepsilon$$

$\|g-\tilde{g}\|_{L^1(X)} < \frac{\varepsilon}{2}$, $\tilde{g}$ cont.

$\forall A \subset X, \mu(A) < \frac{\varepsilon}{2\|\tilde{g}\|_\infty}$

Ex. 51

$\{f_n\}_n \subset L^1(X,\mu)$, $\sup_n \|f_n\|_p < \infty$ for $p \in (1,\infty]$

$\Rightarrow \{f_n\}_n$ satisfy (ii) in equiint. def.:

$$\sup_n \int_A |f_n| \, d\mu \overset{\text{Hölder}}{\le} \underbrace{\|\chi_A\|_q}_{=\mu(A)^{\frac{1}{q}}} \underbrace{\sup_n \|f_n\|_p}_{=:M<\infty}$$

Hence

$$\mu(A) < \left(\frac{\varepsilon}{M}\right)^q \Rightarrow \sup_n \int_A |f_n| < \varepsilon$$

Rem. 52:

a) $\sup_n \|f_n\|_p < \infty, p \in (1,\infty]$ + tightness (i) $\Rightarrow$ equiint.

b) X bnd.: $\sup_n \|f_n\|_p < \infty, p \in (1,\infty] \Rightarrow$ equiint.
((i) ok)

Generally:

Thm. 53 (de la Vallée-Poussin):

$\{f_n\}_n \subset L^1(X,\mu)$, $\mu(X) < \infty$.

$$\{f_n\}_n \text{ equiint.} \iff \sup_n \int \Phi(|f_n|) \, d\mu < \infty$$

for some func. $\Phi \ge 0$, incr., convex, $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$

Lem. 54 (Vitali conv. thm):

(X,μ) meas. sp., $\mu(X)<\infty$, $\{f_n\}$ equiint.,

$f_n(x) \to f(x)$ a.e., $|f(x)|<\infty$ a.e. (f, f_n meas.)

Then $f \in L^1(X)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

Rem. 55:

a) VCT $\rightsquigarrow \lim\limits_{n\to\infty} \int f_n = \int f$

b) VCT $\overset{\mu(X)<\infty}{\Longrightarrow}$ LDCT by Ex. 50

Usefull consequence:

Cor. 56:

$\mu(X)<\infty$, $f_n \in L^1(X,\mu)$, $f_n \to f$ a.e., $\sup\limits_n \|f_n\|_p < \infty$, $p\in(1,\infty]$.

$\Longrightarrow f_n \to f$ in $L^r(X,\mu)$ $\forall\ 1 \le r < p$.

Pf.:

$g_n := |f_n - f|^r$

Obs: $\sup\limits_n \|g_n\|_{\frac{p}{r}} < \infty$, $\frac{p}{r} > 1 \Rightarrow \{g_n\}_n$ equiint.

Since $g_n \to 0$ a.e., by Lem 54,

$0 = \lim\limits_{n\to\infty} \|g_n\|_1 = \lim \|f - f_n\|_r^r$ $\square$

19. W. conv. and comp. in L^1, part II

Alt. char. of w. conv.:

Thm. 57: $\{f_n\}_n \subset L^1(X)$

$$f_n \rightharpoonup f \text{ in } L^1 \iff \begin{cases} \sup_n \|f_n\|_1 < \infty \\ \int_A f_n \, d\mu \to \int_A f \, d\mu \\ \forall A \subset X \text{ meas.} \end{cases}$$

Pf.: ... simple func'ns dense in L^∞ ... $\square$

Rem. 58:

$p = 1$ different from $p > 1$ (Thm 45):

Cannot take $\mu(A) < \infty$, nor A cubes.

Alt. "comp." result

Thm. 58 (Chacon's biting lem.)

$\{f_n\}_n \subset L^1(X)$, $\sup_n \|f_n\|_1 < \infty$.

Then $\exists$ subseq'nce $\{f_{n_k}\}_k$, $f \in L^1(X)$, and $\exists \{\Omega_m\}_m \subset X$, meas., $\Omega_{m+1} \subset \Omega_m$, $\mu(\Omega_m) \to 0$,

s.t.

$$f_{n_k} \rightharpoonup f \text{ on } L^1(X \setminus \Omega_m) \quad \forall m$$

Pf.: Omitted.

Ex. 59:

$f_n = n \chi_{(0, \frac{1}{n})}$: $\sup_n \|f_n\|_1 = 1$ + $f_n \rightharpoonup 0$ in $L^1(\mathbb{R} \setminus [0, k))$, $k > 0$

But $f_n \to \delta_0$ in $\mathcal{M}$

"f in Thm. 58 is the abs. cont. part of the $\mathcal{M}$-lim f_n"

Hence sing. part supported on meas. 0 set.