Sobolevrom 3.4.2017

Sobolev ineq.:

$$1 \le p < d: \quad \|f\|_{L^{p^*}} \overset{\Omega \text{ bnd.}}{\le} C\|f\|_{W^{1,p}}, \quad p^* = \frac{pd}{d-p} \qquad \text{(Ga. Ni. So.)}$$

$$d < p \le \infty: \quad \|f\|_{C^{0,\gamma}} \overset{\Omega \text{ bnd.}}{\le} C\|f\|_{W^{1,p}}, \quad \gamma = 1 - \frac{d}{p} \qquad \text{(Morrey)}$$

11. General Sobolev ineq.'ies

Thm. 35: $\Omega \subset \mathbb{R}^d$, bnd., open, $\partial\Omega$ C^1, $k \in \mathbb{N}_0$, $p \ge 1$,

Assume $f \in W^{k,p}(\Omega)$.

(i) If $k < \frac{d}{p}$, then $f \in L^q(\Omega)$ for

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d},$$

and $\exists\, C = C_{k,p,d,\Omega}$ s.t.

$$\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{k,p}(\Omega)}.$$

(ii) If $k > \frac{d}{p}$, then $f \in C^{k-[\frac{d}{p}]-1,\gamma}(\overline{\Omega})$ for

$$\gamma = \begin{cases} [\frac{d}{p}] + 1 - \frac{d}{p}, & \frac{d}{p} \notin \mathbb{N} \\ \text{any } \gamma \in (0,1), & \frac{d}{p} \in \mathbb{N} \end{cases}$$

and $\exists\, C = C_{k,p,d,\gamma,\Omega}$ s.t.

$$\|f\|_{C^{k-[\frac{d}{p}]-1,\gamma}(\overline{\Omega})} \le C\|f\|_{W^{k,p}(\Omega)}.$$

2)

<u>Rem. 36:</u> $\forall m, p\ \exists k$ s.t. $W^{k,p} \subset C^m$.

$W^{\infty,p} \subset C^\infty \ \forall p$

<u>Pf.:</u>

(i) Since $k < \frac{d}{p}$, $\partial^\alpha f \in L^p(\Omega)\ \forall |\alpha| \le k$

$[\partial^\beta f \in W^{1,p},\ p < \frac{d}{k} < d]$

Ga.Ni.So. $\Longrightarrow$ $\|\partial^\beta f\|_{L^{p^*}(\Omega)} \le C \|f\|_{W^{k,p}(\Omega)}$, $|\beta| \le k-1$.
Thm. 24

$\Longrightarrow f \in W^{k-1,p^*}(\Omega)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$

$\vdots$

$\Longrightarrow f \in W^{k-2,p^{**}}$, $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{d} = \frac{1}{p} - \frac{2}{d}$

$\vdots$

$\Longrightarrow f \in W^{0,q} = L^q$, $\frac{1}{q} = \underbrace{\frac{1}{p} - \frac{k}{d}}_{>0 \text{ since } k < \frac{d}{p}}$

$\|f\|_{L^q} \le \dots \le \tilde{\tilde{C}} \|f\|_{k-2,p^{**}} \le \tilde{C} \|f\|_{k-1,p^*} \le C \|f\|_{k,p}$

(ii) <u>$\frac{d}{p} \notin \mathbb{N}$:</u> By (i)

$f \in W^{k-l,r}$ for $\frac{1}{r} = \frac{1}{p} - \frac{l}{d}$ and $lp < d$.

Let $l = [\frac{d}{p}]$, i.e. $l < \frac{d}{p} < l+1$ and hence

$$r = \frac{pd}{\underbrace{d-pl}_{< p \text{ since } \frac{d}{p} < l+1}} > d.$$

By Morrey's ineq. (Thm. 34),

$$\partial^\alpha f \in C^{0,1-\frac{d}{r}}(\overline{\Omega})\ \forall\ |\alpha| \le k-l-1$$

$$\Downarrow \quad 1 - \frac{d}{r} \overset{\text{def } r}{=} 1 - \frac{d}{p} + l^{=[\frac{d}{p}]}$$

$$f \in C^{k-[\frac{d}{p}]-1,\ [\frac{d}{p}]+1-\frac{d}{p}}(\overline{\Omega})$$

$\frac{d}{p} \in \mathbb{N}$: Let $l = [\frac{d}{p}] - 1 = \frac{d}{p} - 1$. By (i)

$$f \in W^{k-l,r} \quad \text{for} \quad r = \frac{pd}{d-pl} \underset{\text{def. } l}{=} d$$

$\Downarrow$ Ga.Ni.So. (Thm. 24)

$$\partial^\alpha f \in L^q(\Omega) \;\; \forall\, d \le q < \infty,\; |\alpha| \le k - l - \underbrace{1}_{\approx} = k - [\tfrac{d}{p}]$$

$\Downarrow$ Morrey (Thm. 34)

$$\partial^\alpha f \in C^{0,1-\frac{d}{q}}(\overline{\Omega}) \;\; \forall\, d < q < \infty,\; |\alpha| \le k - [\tfrac{d}{p}] \underbrace{-1}_{\approx}$$

$\Downarrow$

$$f \in C^{k-[\frac{d}{k}]-1,\gamma}(\overline{\Omega}) \;\; \forall\, 0 < \gamma < 1 \qquad \square$$

12. Embeddings in Banach sp's

X, Y Banach sp's

$\Phi : X \to Y$

Def. 37

(a) Φ (Banach sp.) embedding if lin., cont., 1-1

(b) Φ cont. embedding, $\Phi : X \hookrightarrow Y$, if

$\exists C > 0$ s.t. $\|\Phi x\|_Y \le C \|x\|_X \quad \forall x \in X$.

(c) Φ comp. emb., $\Phi : X \hookrightarrow\hookrightarrow Y$, if

(i) $\Phi : X \hookrightarrow Y$

(ii) Φ comp., i.e. $\Phi(B)$ precomp. $\forall B \subset X$ bnd.

($\Leftrightarrow \Phi(B)$ seq'al — " ———)

Ex. 39: $\Omega \subset \mathbb{R}^d$ open, bnd; $\partial\Omega$ C^1

id: $W^{1,p}(\Omega) \overset{1\le p<d}{\hookrightarrow} L^{p^*}(\Omega)$ (Ga.-Ni.-So.)

id: $W^{1,p}(\Omega) \overset{p>d}{\hookrightarrow} C^{0,1-\frac{d}{p}}(\bar{\Omega})$ (Morrey)

id: $W^{1,p}(\Omega) \overset{1\le p<d}{\hookrightarrow\hookrightarrow} L^{q}(\Omega)$, $1\le q<p^*$ (Rellich, today)

Rem. 38: need adjoint here!

a) $\Phi(X)$ dense in Y:

$\Phi: X \hookrightarrow Y \underset{\not\Leftarrow}{\Rightarrow} \Phi^*: Y' \hookrightarrow X'$ [HW]

$\Phi: X \hookrightarrow\hookrightarrow Y \underset{\not\Leftarrow}{\Rightarrow} \Phi^*: Y' \hookrightarrow\hookrightarrow X'$

[$\|y\|_{Y'} \overset{\text{dense}}{\underset{\Phi\ \text{lin., bnd.}}{\le}} \|\Phi(x)\|_{X'} + \varepsilon = \sup_{x\neq 0} \frac{|\Phi(x)|}{\|x\|_X} + \varepsilon \overset{X\hookrightarrow Y}{\le} \frac{C\|x\|_X}{\|x\|_X} + \varepsilon = C+\varepsilon$]

Def. 38: X, Y Banach, $X \subset Y$.

a) X cont. emb. in Y if id: $X \hookrightarrow Y$

9:06 b) X comp. emb. in Y if id: $X \hookrightarrow\hookrightarrow Y$

[(i) $\exists C$ s.t. $\|x\|_Y \le C\|x\|_X$ $\forall x \in X$;

(ii) $\forall \{x_n\} \subset X$ bnd. $\Rightarrow \exists \{x_{n_k}\}_k$ Cauchy in Y]

Lem. 39: $X \subset Y \subset Z$ normed sp's, $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$.

Then $\forall \varepsilon>0$ $\exists C_\varepsilon>0$ s.t.

$$\|x\|_Y \le \varepsilon\|x\|_X + C_\varepsilon\|x\|_Z. \quad \forall x\in X.$$

Pf.: HW, see Holden p. 19

Ex. 40: $\underbrace{W^{2,p}}_{\Omega\ \text{bnd.}} \hookrightarrow\hookrightarrow W^{1,p} \hookrightarrow\hookrightarrow L^p \overset{\text{Lem. 39}}{\Longrightarrow} \|\nabla f\|_p \le \varepsilon\|\partial^2 f\|_p + C_\varepsilon\|f\|_p$

Str.

13. Compactness in $W^{1,p}(\Omega)$

Bnd. subsets of $W^{1,p}$ are precomp. in other sp's:

Thm. 41: (Rellich-Kondrachov comp. thm.)

$\Omega \subset \mathbb{R}^d$, bnd, open; $\partial\Omega$ C^1.

(a) $1 \le p < d$: $W^{1,p}(\Omega) \underset{\text{comp. emb.}}{\hookrightarrow\hookrightarrow} L^q \quad \forall\, q \in [1, p^*)$

(b) $p = d$: $W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L^q \quad \forall\, q \in [1, \infty)$

(c) $p > d$: $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma}(\bar{\Omega}) \quad \forall\, \gamma \in (0, 1-\frac{d}{p})$

Obs. 42:

$\sup_n \|f_n\|_{W^{1,p}(\Omega)} < \infty \Rightarrow \exists\, f_{n_k}, \tilde{f}$ s.t.

$$f_{n_k} \longrightarrow \tilde{f} \text{ in } \begin{cases} L^q & 1 \le p < d \\ C^{0,\gamma} & p > d \end{cases}$$

Arzela-Ascoli (C^0) $\rightsquigarrow$ Kolmogorov-Riesz (L^p) $\rightsquigarrow$ Rellich-Kondrachov ($W^{1,p}$)

Pf.:

(b) Follows from (a) since $W^{1,p} \underset{p<d}{\subset} W^{1,d}$ and $p^* \underset{p\to d}{\longrightarrow} \infty$.

(c) By Morrey (Thm 34), $W^{1,p} \hookrightarrow C^{0,1-\frac{d}{p}}$.

By $\underbrace{\text{Arzela-Ascoli}}_{\text{(Thm. 49, part 1)}}$, $C^{0,\gamma} \underset{0<\gamma'<\gamma}{\hookrightarrow\hookrightarrow} C^{0,\gamma'}$ [HW],

and (c) follows.

(a) $(1 \le p < d)$

1) Cont. emb.: $W^{1,p} \hookrightarrow L^{p^*} \hookrightarrow L^q$, $q \in [1, p^*)$

$\uparrow$ Ga.-Ni.-So. Thm 24 $\quad \uparrow$ Ω bnd.

2) Comp. 1: $W^{1,p} \hookrightarrow\hookrightarrow L^p$

$\{f_n\}_n \subset W^{1,p}(\Omega)$ bnd

$\Downarrow$ extension (Thm. 15)

$\{\bar{f}_n\}_n \subset W^{1,p}(\mathbb{R}^d)$ bnd., $\operatorname{supp} \bar{f}_n \subset B$ bnd.

$(\Omega \subset B)$

$\Downarrow$

i) $\sup_n \|\bar{f}_n\|_p < \infty$

ii) $\sup_n \|\bar{f}_n - \tau_h \bar{f}_n\|_p^p = \sup_n \int |\underbrace{f_n(x) - f_n(x+th)}_{\le |h| \int_0^1 |\nabla f(x+th)| dt}|^p dx$

Jensen

$\le |h|^p \sup_n \iint_0^1 |\nabla \bar{f}_n(x+th)|^p dt\, dx \underset{\text{Fubini}}{=} |h|^p \underbrace{\sup_n \|\nabla f_n\|_p^p}_{<\infty}$

iii) $\sup_n \int_{\mathbb{R}^d \setminus B} |\bar{f}_n|^p = 0$

$\Downarrow$ Kolmogorov comp. thm (Thm 19, part 3)

$\{\bar{f}_n\}$ precomp. in $L^p(\mathbb{R}^d)$

3) Comp. 2: $W^{1,p} \hookrightarrow\hookrightarrow L^q$, $q \in [1, p^*)$

Ga.-Ni.-So (Thm 24) $\Rightarrow \sup_n \|\bar{f}_n\|_{p^*} \le C \underset{\sup_n}{\uparrow} \|\bar{f}_n\|_{1,p} \overset{\text{ext.}}{\le} \underbrace{\tilde{C} \sup \|f_n\|_{1,p}}_{<\infty} =: K$

By 2), $\exists\, f_{n_k}$ Cauchy in L^p, and
by interpolation (Prop. 9, part 3):

$$\|\bar f_{n_k} - \bar f_{n_l}\|_q \underset{\substack{\uparrow \\ \frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*},\ \theta \in [0,1]}}{\le} \|\bar f_{n_k} - \bar f_{n_l}\|_1^{\theta} \underbrace{\|\bar f_{n_k} - \bar f_{n_l}\|_{p^*}^{1-\theta}}_{\le (2K)^{1-\theta}}$$

$$\underset{\substack{\Omega \text{ bnd} \\ p \ge 1}}{\le} (2K)^{1-\theta} C \|\bar f_{n_k} - \bar f_{n_l}\|_p^{\theta}$$

$\Rightarrow \{\bar f_{n_k}\}$ Cauchy in L^q

$\Rightarrow \{\bar f_n\}$ precomp. in $L^q(\mathbb{R}^d)$

$\Rightarrow \{f_n\}$ precomp. in $L^q(\Omega)$ $\square$