Sobolevrom 6.4.2017

13. Str. comp. on $W^{1,p}(\Omega)$ (cont.)

Rellich – Kondrachov comp. thm. (Thm. 41):

$\Omega \subset \mathbb{R}^d$ open, bnd, $\partial\Omega$ C^1

$$W^{1,p}(\Omega) \underset{\text{comp. emb.}}{\hookrightarrow\hookrightarrow} \begin{cases} L^q(\Omega),\ q\in[1,p^*) & \text{if } 1\le p<d \\ C^{0,\gamma}(\bar\Omega),\ \gamma\in(0,1-\frac{d}{p}) & \text{if } p>d \end{cases}$$

Pf.: Kolmogorov / A.-A. + Sobolev / Morrey ineq. + $L^p/C^{0,\gamma}$ interpolation

$p=\infty$: A.-A. □

Rem. 43:

a) 1D: $W^{1,p}(\overset{\text{open interval}}{I}) \hookrightarrow\hookrightarrow \begin{cases} C^{0,\gamma},\ \gamma\in(0,1-\frac{1}{p}) & \text{if } p>1 \\ L^q,\ q\in[1,\infty) & \text{if } p=1 \end{cases}$

[$p>1$: ok by Thm. 41.
$p=1$: Ext. + Komogorov in L^1 + L^∞_{bnd} (1D Sob. ineq.) + interp.]

b) Same arg. as Thm 41

$\Downarrow$

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q,\ q\in[1,p^*) & \text{if } 1\le p<d \\ C^{0,\gamma},\ \gamma\in(0,1-\frac{d}{p}) & \text{if } p>d \end{cases}$$

open, bnd, not cond'n on $\partial\Omega$!

c) Generally:

$$W^{k,p}(\Omega)/W_0^{k,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q,\ q\in[1,\frac{dp}{d-kp}) & ,\ k<\frac{d}{p} \\ C^{k-[\frac{d}{p}]-1,\bar{\gamma}},\ \bar{\gamma}\in(0,\gamma) & ,\ k>\frac{d}{p} \end{cases}$$

need $\partial\Omega$ C^k

from gen. Sob. ineq.

d) $W^{1,p}(\Omega) \overset{\Omega \text{ bnd...}}{\hookrightarrow\hookrightarrow} L^p(\Omega)$ $\forall p\geq 1$ $\forall d\in\mathbb{N}$ $[L^\infty \subset L^{p^*}\subset L^p]$

14. Sobolev chain rule

$$(*) \quad \partial_i[F(f)] = \nabla F(f)\cdot\partial_i f$$

Thm. 44: $\Omega\subset\mathbb{R}^d$ open, $p\geq 1$, $F\in C^1(\mathbb{R}^n)$, $\nabla F\in L^\infty(\mathbb{R}^n)$, $f=(f_1,\dots,f_n)$.

a) If $f\in(W^{1,p}_{loc}(\Omega))^n$, then $F(f)\in W^{1,p}_{loc}(\Omega)$ and $(*)$ holds in D'!

~~$(*)$ $\partial_j[F(f)] = \nabla F(f)\cdot\partial_j f.$~~

b) If $f\in(W^{1,p}(\Omega))^n$ and $|\Omega|<\infty$ or $F(0)=0$, then $F(f)\in W^{1,p}(\Omega)$ and $(*)$ holds in D'

Pf.:

1) $(*)$ holds in D':

By approx. $\exists\ C^1(\Omega)\ni f_m \longrightarrow f$ in $W^{1,p}_{loc}$ and a.e. (subseq'nce)

Chain rule for $F(f_m)$:

$$\partial_i[F(f_m)] = \nabla F(f_m)\cdot\partial_i f_m$$

$\Downarrow \cdot \psi \in C_c^\infty + \int dx + $ i.b.p.

$$-\int F(f_m)\,\psi_{x_i} = \int \nabla F(f_m)\,\partial_i f_m\,\psi$$

$$= \int \underbrace{\nabla F(f_m)\,\partial_i f\,\psi}_{\le \|\nabla F\|_\infty \|\psi\|_\infty |\partial_i f| \chi_{\mathrm{supp}\,\psi} \in L^1}$$

$$+ \underbrace{\int \nabla F(f_m)(\partial_i f_m - \partial_i f)\psi}_{\le \|\nabla F\|_\infty \|\psi\|_\infty \underbrace{\|f_m - f\|_{1,1,\mathrm{supp}\,\psi}}_{\le C\|f_m - f\|_{1,p,\mathrm{supp}\,\psi} \longrightarrow 0}}$$

LDCT $\Big\downarrow$ $F(f_m) \to F(f)$ a.e., $\nabla F(f_m) \to \nabla F(f)$ a.e.

$$-\int F(f)\,\psi_{x_i} = \int \nabla F(f)\cdot \partial_i f\,\psi$$

2) $f \in (W^{1,p}_{loc})^n / (W^{1,p})^n,\ \nabla F \in L^\infty \overset{(*)}{\Longrightarrow} \nabla F(f) \in L^p_{loc}/L^p$

3) $|F(f)| \le |F(0)| + \|\nabla F\|_\infty |f|$, hence

$$f \in (L^p_{loc})^n / f \in (L^p)^n \underset{\substack{+\,|F(0)|=0 \\ \text{if } |\Omega|=\infty}}{} \Rightarrow F(f) \in L^p_{loc}/L^p \qquad \square$$

Some important func'ns are not C^1:

$f^+ = \max(f,0)$, $f^- = -\min(f,0)$, $|f| = f^+ + f^-$

Lem. 45: $\Omega \subset \mathbb{R}^d$ open, $p \geq 1$, $f \in W^{1,p}(\Omega)$.

Then $f^+, f^-, |f| \in W^{1,p}(\Omega)$ and

$$\nabla f^+ = \begin{cases} \nabla f & \text{where } f > 0 \\ 0 & \text{"} \quad f \leq 0 \end{cases}$$

$$\nabla f^- = \begin{cases} 0 & \text{"} \quad f \geq 0 \\ -\nabla f & \text{"} \quad f < 0 \end{cases}$$

$$\nabla |f| = \begin{cases} \nabla f & \text{"} \quad f > 0 \\ 0 & \text{"} \quad f = 0 \\ -\nabla f & \text{"} \quad f < 0 \end{cases}$$

Pause
14:59

Pf.:

Let $\varepsilon > 0$, def.

$$G_\varepsilon(f) = \begin{cases} \sqrt{f^2 + \varepsilon^2} - \varepsilon & \text{where } f > 0 \\ 0 & \text{"} \quad f \leq 0 \end{cases}$$

By Thm. 44, $\forall \psi \in C_c^\infty(\Omega)$

$$\int_\Omega G_\varepsilon(f) \nabla \psi = - \int_{f>0} \frac{f \nabla f}{\sqrt{f^2+\varepsilon^2}} \psi$$

$$\downarrow \qquad \downarrow \varepsilon \to 0$$

$$\int_\Omega f^+ \nabla \psi = - \int_{f>0} \nabla f \, \psi$$

$$\Downarrow$$

f^+ res.

$\Downarrow$ $f^- = (-f)^+$

f^- res.

$\Rightarrow |f| = f^+ + f^-$ res.

$\square$

15. Difference quotients / finite differences

$f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, $h > 0$

Difference op.: $D_{\pm,i} = D^h_{\pm,i}$, $D_\pm = (D_{\pm,1}, \dots, D_{\pm,d})$

$$D_{\pm,i} f(x) = \pm \frac{f(x \pm h e_i) - f(x)}{h} \quad \text{(forw. / backw. diff.)}$$

Obs. 46:

$$D_{\pm,i} f(x) \in L^1_{loc}(\Omega') \quad \text{if} \quad 0 < h < dist(\Omega', \partial\Omega)$$

Thm. 47:

(a) $1 \leq p < \infty$, $\overline{\Omega'} \subset \Omega$. If $f \in W^{1,p}(\Omega)$,

then $\exists\, C > 0$ s.t. $\forall\, 0 < h < \frac{1}{2} dist(\Omega', \partial\Omega)$

$$\| D_\pm f \|_{L^p(\Omega')} \leq C \| \nabla f \|_{L^p(\Omega)}$$

(b) $1 < p < \infty$, $\overline{\Omega'} \subset \Omega$. If $f \in L^p(\Omega)$ and

$\exists\, C > 0$ s.t. $\forall\, 0 < h < \frac{1}{2} dist(\Omega', \partial\Omega)$

$$\| D_\pm f \|_{L^p(\Omega')} \leq C,$$

then $f \in W^{1,p}(\Omega')$ and

$$\| \nabla f \|_{L^p(\Omega')} \leq C.$$

Rem. 48:

a) Part (b) false when $p = 1$.

b) Usefull to prove regularity of sol'ns of PDEs:

Ex. 49: (HW)

(**) $u - \Delta u = f$ in $\mathbb{R}^d$, $f \in L^2(\mathbb{R}^d)$

Thm.: $\exists !$ w. sol'n $u \in W^{1,2}(\mathbb{R}^d)$ of (**), i.e.

$$\underbrace{\int u\psi + \int \nabla u \cdot \nabla \psi}_{(u,\psi)_{W^{1,2}}} = \int f \cdot \psi \quad \forall \psi \in W^{1,2}(\mathbb{R}^d)$$

[Riesz-repr. thm.]

Lem.: $\|u\|^2_{W^{1,2}} = \left|\int f u\right| \overset{F(\psi):=\int f\psi}{=} \|u\|_{W^{1,2}} \frac{|F(u)|}{\|u\|_{W^{1,2}}} \le \|u\|_{W^{1,2}} \underbrace{\sup_{0 \neq \varphi \in W^{1,2}} \frac{|F(\varphi)|}{\|\varphi\|_{W^{1,2}}}}_{\|F\|_{(W^{1,2})'}}$

Chk.: $D_+ u$ w. sol'n of (**) w. R.H.S. $D_+ f$

Chk.: $\|D^+ f\|_{(W^{1,2})'} \le C \|f\|_{L^2}$

Conclude $u \in W^{2,2}$ and $\|u\|_{W^{2,2}} \le C \|f\|_{L^2}$.

Pf.:

(a) Approx.: $\exists\, C^1(\Omega) \ni f_m \longrightarrow f$ in $W^{1,p}_{loc}$ and a.e. (subseq'nce)

$$\int_{\Omega'} |\underbrace{D_{\pm,i} f_m}|^p \overset{\text{Jensen}}{\le} \int_{\Omega'} \int_0^1 |\partial_i f_m(x + t h e_i)|^p \, dt\, dx \overset{\text{Fubini}}{\le} \|\partial_i f_m\|^p_{L^p}$$

$$= \frac{1}{h} \int_0^1 \partial_i f(x \pm t h e_i)\, dt \cdot h$$

$m \to \infty \implies$ (a) [Use Fatou on L.H.S.]

(b) Let $\varphi \in C_c^\infty(\Omega')$.

~~Chk.:~~ $\cancel{\int_{\Omega'} D_{\pm,i} f \varphi = -\int_{\Omega'} f D_{\mp,i}\varphi}$

Since $p \in (1,\infty)$ and $\|D^h_\pm f\|_{L^p(\Omega')} \le C \;\forall h$,

$\exists\, D^{h_k}_{\pm,i} f,\ f_i \in L^p$ s.t. $D^{h_k}_{\pm,i} f \xrightarrow[k\to\infty]{L^p} f_i$ in $L^p(\Omega')$

[Eberlein-Smulian]

Chk.: $\int_{\Omega'} D^{h_k}_{\pm,i} f \varphi = - \int_{\Omega'} f D_{\mp,i} \varphi$

$$\downarrow \; k \to \infty$$

$$\int_{\Omega'} f_i \varphi = - \int_{\Omega'} f \varphi_{x_i}$$

$$\Rightarrow \quad f_{x_i} \overset{D'}{=} f_i \in L^p \quad \Rightarrow f \in W^{1,p} \qquad \square$$