Sobolevrom 9.3.2017

22. Radon meas., $\mathcal{M} = (C_0)'$

Ref.: Folland: Real anal. chp. 7 ; Holden

X loc. comp. Hausdorff (LCH) sp. (very general - see Holden)

Def. 66: a) Radon meas. μ: A pos. Borel meas. on X satisfying

(i) $\mu(K) < \infty \quad \forall K \subset X$ comp. (loc. finite)

(ii) $\mu(B) = \inf\{\mu(U) : U \supset B, U \text{ open}\} \; \forall B \subset X$ Borel

(outer reg.)

(iii) $\mu(U) = \sup\{\mu(K) : K \subset U \text{ comp.}\} \; \forall U \subset X$ open

(inner reg.)

b) Signed Radon meas. μ: $\mu = \mu^+ - \mu^-$ ("Jordan decomp.")

and $\mu^\pm$ Radon meas.

c) $\mathcal{M}(X) := \{\mu : \mu \text{ signed Radon m.}, \; \underbrace{|\mu|(X)}_{\text{tot. var.}} := (\mu^+ + \mu^-)(X) < \infty\}$

sp. of finite signed Radon meas.

d) $\mathcal{M}^+(X) := \{\mu : \mu \in \mathcal{M}, \mu \geq 0\}$

Rem. 67:

a) Radon meas.: dx (Lebesgue); $f(x)\,dx$, $f \in L^1_{loc}$; δ-meas.

Not: δ'

b) $|f\,dx|(B) = \int_B |f|\,dx \quad \forall B \subset X$ meas., $f \in L^1$

$\left|\int_B f\,d\mu\right| \le \int_B |f|\,d|\mu|$, $f \in L^1(B, |\mu|)$

$\mu \geqslant 0 \implies |\mu| = \mu$

μ (signed) Radon $\iff$ $|\mu|$ Radon

c) $(\mathcal{M}, |\cdot|(X))$ Banach sp. [Normed sp. Follan Prop 7.16 Complete by $\simeq C_0(X)'$ which is compl.]

Lin. func'als def. by Radon meas.:

$$\Phi_\mu(f) = \int f\,d\mu, \quad \mu \text{ Radon};$$

i) $\Phi_\mu(f)$ well-def., finite for $f \in C_c(X)$ (cont. w. comp. supp)

ii) Φ_μ lin.; $\|\Phi_\mu\| = \sup\limits_{\substack{f \in C_c \\ \|f\|_\infty \le 1}} \left|\int_X f\,d\mu\right| < \infty \iff |\mu|(X) < \infty$

$\left[\, \left|\int_X f\,d\mu\right| \le \|f\|_\infty |\mu|(X) \,\right]$

Hence $\Phi_\mu \in (C_c)'$ iff $|\mu|(X) < \infty$.

Def. 68: $C_0(X) := \overline{C_c(X)}^{\|\cdot\|_\infty} = \{ f \in C(X) : \exists f_n \in C_c,\ \|f_n - f\|_\infty \to 0 \}$

Rem. 69:

(a) $C_0(X) = \{ f \in C(X) : \forall \varepsilon > 0\ \exists K_\varepsilon \subset X$ comp. (bnd!) s.t. $y \in X \setminus K_\varepsilon \implies |f(y)| < \varepsilon \}$

$C_0(X) \subset C_b(X)$ "f vanish ~~at~~ at ∞"

(b) $(C_0, \|\cdot\|_\infty)$ Banach sp. Separable if $X \subset \mathbb{R}^d$.

(c) X comp. $\Rightarrow$ $C(X) = C_c(X) = C_0(X)$.

$|\mu|(X) < \infty$ $(\Rightarrow \mu \in \mathcal{M})$: Φ_μ well-def. for $f \in C_0$. [same reason as for $f \in C_c$]

~~$\Phi_\mu(f) = \lim_{n\to\infty} \Phi_\mu(f_n)$ where $\|f_n - f\|_\infty \to 0$, $f_n \in C_c$~~

~~$[\, |\Phi_\mu(f_n) - \Phi_\mu(f_k)| \le \|\Phi_\mu\| \, \|f_n - f_k\|_\infty$ Cauchy in $\mathbb{R}$ ...]~~

Hence $\Phi_\mu \in (C_0)'$ $\forall$ $\mu \in \mathcal{M}$, and

$$\mathcal{M} \subset (C_0)' \quad (\Phi(\mathcal{M}) \subset (C_0)')$$

The opposite is also true:

<u>Thm. 70:</u> "Riesz (-Markov) repr. thm.

X LCH, $\Phi : \mathcal{M}(X) \to C_0(X)'$, $\mu \mapsto \Phi_\mu$.

Then Φ isometric isomorphism from $\mathcal{M}$ to $C_0(X)'$.

<u>Pf.</u>: Omitted.

I.e. Φ invertible, lin., and

$$\|\mu\|_{\mathcal{M}} := |\mu|(X) = \|\Phi_\mu\| = \sup_{\substack{f \in C_0 \\ \|f\|_\infty \le 1}} \left| \int f \, d\mu \right|,$$

Obs: $\left| \int f \, d\mu \right| \le \|\mu\|_{\mathcal{M}} \|f\|_\infty \quad \forall f \in C_0$

We identify $\mathcal{M}$ and $C_0(X)'$:

$$\mathcal{M} = (C_0)' \quad (\text{really } \Phi(\mathcal{M}) = (C_0)')$$

W. * conv. in $\mathcal{M}$:

$$\mu_n \xrightarrow{*} \mu \text{ in } \mathcal{M} \overset{\text{def.}}{\Longleftrightarrow} \int_X f\,d\mu_n \to \int_X f\,d\mu \ \forall f \in C_0$$

Prop. 71: $X \subset \mathbb{R}^n$; $\mu_n, \mu \geq 0$.

[Not true for $|\mu_n|$
[$\mu_k = \delta_{\frac{1}{k}} - \delta_{-\frac{1}{k}}$
$\mu_k \xrightarrow{*} 0$ in $\mathcal{M}$
$|\mu_k| = \delta_{\frac{1}{k}} + \delta_{-\frac{1}{k}}$]

If $\mu_n \xrightarrow{*} \mu$ in $\mathcal{M}$, then

(i) $\limsup_n \mu_n(K) \leq \mu(K) \ \forall K \subset X$ comp.

(ii) $\mu(U) \leq \liminf_n \mu_n(U) \ \forall U \subset X$ open

[(ii): $K \subset U$ comp., take f, $0 \leq f \leq 1$, $=1$ in K, $=0$ in U^c: $\chi_K \leq f \leq \chi_U$.
$\mu(K) = \int \chi_K \, d|\mu| \leq \int f\,d\mu \underset{\mu_n \xrightarrow{*} \mu}{=} \lim \int f\,d\mu_k = \liminf - \leq \liminf_k \int \chi_U \, d\mu_k = \liminf_k \mu_k(U)$
$\mu(U) \overset{\mu \text{ Radon}}{=} \sup\{\mu(K) : K \subset U \text{ comp}\} \leq \liminf \mu_k(U)$]

Ex. 72:

a) $f_n = n\,\chi_{(0,\frac{1}{n})} \in L^1(0,1)$ $[\|f_n\|_1 = 1]$
$X = [0,1]$
$\mu_n = f_n\,dx \in \mathcal{M}$

$$\int g\,d\mu_n = \int g f_n\,dx \to g(0) \cdot 1 \quad \forall g \in C([0,1]) = C_0([0,1])$$

$$\Rightarrow \mu_n \xrightarrow{*} \delta_0 \text{ in } \mathcal{M}$$

But: $|\mu_n - \delta_0|(X) = (\mu_n - \delta_0)^+(X) + (\mu_n - \delta_0)^-(X) = \mu_n(X) + \delta_0(X) = 2$

HW: a) $f_n \to f$ in L^1 $\not\Rightarrow$ $\mu_n = f_n\,dx \xrightarrow{*} \mu = f\,dx$ (HW?)

5:05

Rem. 73: $\|\mu_n - \mu\|_{\mathcal{M}(X)} \to 0 \underset{\not\Leftarrow}{\Rightarrow} \mu_n \xrightarrow{*} \mu$ in $\mathcal{M}$

[$\Rightarrow$) $|\int g\,d(\mu - \mu_n)| \leq \|g\|_\infty \|\mu - \mu_n\|_{\mathcal{M}(X)}$, $\not\Leftarrow$) Ex. 72]

23. W.* seq'al comp. for Radon meas.

$X = \mathbb{R}^d$

Thm. 74 ("w.*" seq'al comp.)

μ_n, $n \in \mathbb{N}$, Radon meas. on $\mathbb{R}^d$ s.t.

$$\sup_n |\mu_n|(K) < \infty \quad \forall\ K \subset \mathbb{R}^d \text{ comp.} \qquad [\text{or } \mu_n, \mu \geq 0!$$

Then $\exists$ subseq'nce μ_{n_k}, Radon meas. μ s.t.

$$\int g\, d\mu_{n_k} \to \int g\, d\mu \ \forall\ g \in C_c(\mathbb{R}^d).$$

Thm. 75 (w.* seq'al comp. on $\mathcal{M}$)

$$\mu_n \in \mathcal{M}(\mathbb{R}^d), \quad \sup_n \|\mu_n\|_{\mathcal{M}} < \infty$$

$$\Rightarrow \exists\ \mu_{n_k}, \mu \in \mathcal{M} \text{ s.t. } \mu_{n_k} \xrightarrow{*} \mu \text{ in } \mathcal{M}$$

Comp. in $L^1(\mathbb{R}^d, \nu)$:

$$L^1 \subset \mathcal{M};\ L^1 \ni f \mapsto f\nu \in \mathcal{M};$$

$$\sup_n \|f_n\|_1 < \infty \Rightarrow \sup \underbrace{\|f_n \nu\|_{\mathcal{M}}}_{= \sup_{\substack{g \in C_0 \\ \|g\|_\infty \leq 1}} |\int g f_n\, d\nu| \leq 1 \cdot \|f_n\|_1} < \infty$$

By Thm 75:

Cor. 76:

$$\{f_n\} \subset L^1, \quad \sup_n \|f_n\|_1 < \infty$$

$$\Rightarrow \exists\ f_{n_k}, \mu \in \mathcal{M} \text{ s.t.}$$

$$\int g f_{n_k}\, d\nu \to \int g\, d\mu \quad \forall\ g \in C_0(\mathbb{R}^d)$$

Obs: Thm. 74 $\overset{\Rightarrow}{\Leftarrow}$ Thm 75

[$\Rightarrow$ [Take $C_c \ni f_j \xrightarrow{L^\infty} f \in C_0$ + use $\sup_n |\mu_n|(\mathbb{R}^d) < \infty$ ...] Easy - HW

[$\Leftarrow$ Consider $\mu_n^m(B) = \mu_n(B \cap B(0,m))$.

Thm. 75 + diag. arg. $\Rightarrow \exists\, \mu_{n_k}^{m_k}, \mu$ s.t.

$$\int g \, d\mu_{n_k}^{m_k} \to \int g \, d\mu \quad \forall\, g \in \underset{C_0(\overline{B(0,m)})}{\overset{\|}{C(\overline{B(0,m)})}} \ \forall m$$

Chk.: g def. Radon meas. ($g|_{B(0,m)} \in \mathcal{M}$ by Riesz)]

Pf. of Thm. 75:

1) Candidate limit:

Take $\{f_j\}_j \subset C_0(\mathbb{R}^d)$ dense (C_0 separable)

By assumption:

$$\left| \int f_j \, d\mu_n \right| \le \underbrace{\sup_n \|\mu_n\|_{\mathcal{M}}}_{=: M < \infty} \|f_j\|_\infty ,$$

hence $\exists\, n_k, \alpha_j \in \mathbb{R}$ s.t. (comp. in $\mathbb{R}$)

$$\int f_j \, d\mu_{n_k} \xrightarrow[k\to\infty]{} \alpha_j$$

Diag. arg. gives further subseq'nce n_l s.t. and $\alpha_j, j \in \mathbb{N}$

$$\int f_j \, d\mu_{n_l} \xrightarrow[l\to\infty]{} \alpha_j \quad \forall j$$

Def. $\Lambda(f_j) = \alpha_j \ \forall j$ and

$$\Lambda(f) = \lim_{\|f_j - f\|_\infty \to 0} \Lambda(f_j) \quad \forall\, f \in C_0 = \overline{\{f_j\}_j}^{\|\cdot\|_\infty}$$

Chk.: $\Lambda \in (C_0)'$ [$\|\Lambda\| \le M$]

$\overset{\text{Riesz}}{\Rightarrow} \exists\, \mu \in \mathcal{M}$ s.t. $\Lambda(f) = \int f \, d\mu \ \forall\, f \in C_0$

[$\mu = \Phi^{-1}\Lambda$]

2) Convergence:

Let $f \in C_0$ and $\{\tilde{f}_j\}_j \subset \{f_j\}_j$ s.t. $\|\tilde{f}_j - f\|_\infty \to 0$.

Take J s.t. $\|f - \tilde{f}_j\|_\infty < \frac{\varepsilon}{M}$ for $j \geq J$,

N s.t. $|\int \tilde{f}_J \, d\mu_n - \int \tilde{f}_J \, d\mu| < \varepsilon$ for $n \geq N$.

Hence

$$|\int f \, d\mu_n - \int f \, d\mu|$$

$$\leq \underbrace{|\int (f - \tilde{f}_J) \, d\mu_n|}_{< \frac{\varepsilon}{M} \cdot M} + \underbrace{|\int \tilde{f}_J \, d\mu_n - \int \tilde{f}_J \, d\mu|}_{< \varepsilon} + \underbrace{|\int (\tilde{f}_J - f) \, d\mu|}_{< \frac{\varepsilon}{M} \cdot M}$$

$$< 3\varepsilon$$

$\square$