Sobolevrom 16. 3. 2017

HW: $u \in C_c^\infty$, $v \in W^{m,p}$
$\Rightarrow u \cdot v \in W^{m,p}$
$\|u \cdot v\|_{m,p} \le c \|u\|_{m,\infty} \|v\|_{m,p}$

Next year:

Lem: $\Phi : \Omega \xrightarrow{C^m} G$, $\Phi^{-1} : G \xrightarrow{C^m} \Omega$

$\Rightarrow W^{m,p}(\Omega) \xrightarrow{\Phi} W^{m,p}(G)$

$u \mapsto u \circ \Phi^{-1}$

HW before Lem 12
($m = 1$)

Last time:

$\Omega \subset \mathbb{R}^d$ open, $m \in \mathbb{N}$, $p \in [1, \infty]$

$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \underset{\text{w. deriv.}}{\partial^\alpha f} \in L^p(\Omega), \ \forall |\alpha| \le m \}$

$$\|f\|_{m,p} = \Big(\sum_{|\alpha| \le m} \|\partial^\alpha f\|_p^p \Big)^{\frac{1}{p}}$$

2. Smooth approx. (cont.)

A. Interior approx. (Thm 6): $C^m(\Omega) \ni f_n \xrightarrow{W^{m,p}_{loc}} f \in W^{m,p}(\Omega)$

B. Global approx. (Thm 7): $C^m(\Omega) \ni f_n \xrightarrow[(\Omega \text{ bnd})]{W^{m,p}} f \in W^{m,p}(\Omega)$

C. Global up to the bnd'ry approx. by smooth func'n

Approx. by $f_n \in C^m(\overline{\Omega})$.

Need cond'ns on $\partial\Omega$ ("nice")

Thm. 8: $\Omega \subset \mathbb{R}^d$ open, bnd; $\partial\Omega$ is C^1; $f \in W^{m,p}(\Omega)$.

Then $\exists f_n \in C^\infty(\overline{\Omega})$ s.t. $\|f - f_n\|_{m,p} \xrightarrow[n\to\infty]{} 0$.

~~**Tool 1:** Part. of 1 (last time)~~

~~**Tool 2:** Straightening the bnd'ry~~

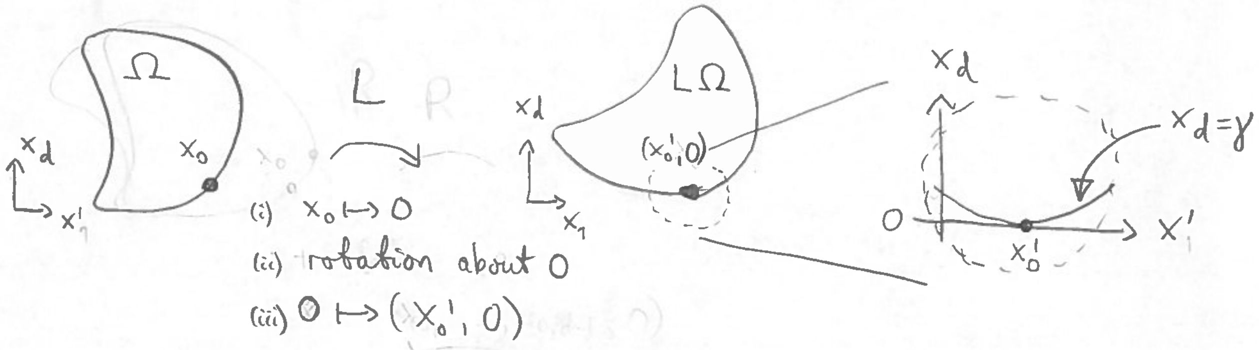
rot + straigh + perm not vol changing

Def. 9: $\Omega \subset \mathbb{R}^d$ is C^k ($\partial\Omega$ is C^k) if

$\forall x_0 \in \partial\Omega \quad \exists r > 0,\ \gamma \in C^k(\mathbb{R}^{d-1})$, L transl./rot. ($\det DL = 1$)
transl. + rotation

s.t.

$$\underset{\{Lx : x \in \Omega\}}{\underset{\parallel}{L\Omega}} \cap B((x_0',0),r) = \{x = (x_1, \dots, x_d) \mid x_d > \gamma(\underbrace{x_1, \dots, x_{d-1}}_{x'})\}$$
$$= (x', x_d) \in B((x_0',0),r)$$



Obs. 10:

rotation (matr.): $RR^T = I$

a) $Lx = R(x - x_0) + (x_0', 0)$

$L^{-1}y = R^{-1}(y - (x_0', 0)) + x_0$

$LB(x_0, r) = B((x_0', 0), r)$

b) L, L^{-1} vol. preserving affine coord. transf.:

$D(Lx) = R,\ D(L^{-1}y) = R^{-1},\ \det(DLx) = \det R = 1 = \det R^{-1} = \det(DL^{-1}y)$

c) $v(x) = u(L^{-1}x) \overset{\text{chk}}{\underset{\text{b)}}{\Rightarrow}} \|v\|_{L\Omega,m,p} = C_{R,m,p}\|u\|_{\Omega,m,p}$

$+\ \nabla v = R^{-1} \cdot \nabla u \cdots$

Pf. of Thm 8: (Evans: PDEs)

1) Fix $x_0 \in \partial\Omega$.

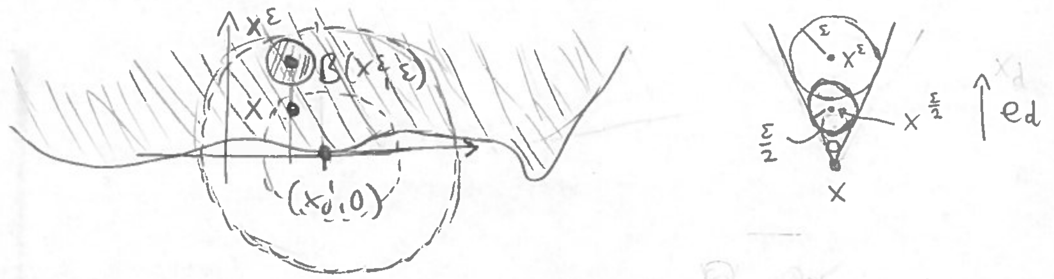
$\partial\Omega\ C^1 \Rightarrow \exists\, r>0,\ \gamma \in C^1(\mathbb{R}^{d-1}),\ L$ s.t.

$\tilde{\Omega}_r := L\Omega \cap B((x_0',0),r) = \{x \in B(-,r) : x_d > \gamma(x')\}$

2) Def.: $x^\varepsilon = x + \lambda\varepsilon e_d$, $x \in \tilde{\Omega}_{\frac{r}{2}}$, $\varepsilon > 0$.

$\exists\, \lambda > 0$ big enough s.t.

(*) $B(x^\varepsilon, \varepsilon) \subset \tilde{\Omega}_r\ \forall\, x \in \tilde{\Omega}_{\frac{r}{2}}$, $\varepsilon > 0$ small



[Ok since "interior cone cond'n" holds in C^1 (or Lip) domains:

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3) Def.: $g(x) := f(L^{-1}x)$, $g^\varepsilon(x) := g(x^\varepsilon)$, and

$$g_\varepsilon(x) := g^\varepsilon * \rho_\varepsilon(x) \quad \text{for } x \in \tilde{\Omega}_{\frac{r}{2}},$$

where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, $0 \le \rho \in C_c^\infty$, $\int\rho = 1$, $\operatorname{supp}\rho \subset B(0,1)$.

Def.: $f_\varepsilon(x) := g_\varepsilon(Lx)$, $x \in \Omega_{\frac{r}{2}} := \Omega \cap B(x_0, \frac{r}{2})$

OBS: By (*), $g_\varepsilon|_{\tilde{\Omega}_{\frac{r}{2}}}$ dep. only on $g|_{\tilde{\Omega}_r}$,

$\underset{L\Omega_r = \tilde{\Omega}_r}{\overset{\text{Obs. 10}}{\Longrightarrow}}$ $f_\varepsilon|_{\Omega_{\frac{r}{2}}}$ dep. only on $f|_{\Omega_r}$

15:09

Chk.: $f_\varepsilon \in C^\infty(\overline{\Omega_{\frac{r}{2}}})$ (easy)

$$\|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{\Omega_{\frac{r}{2}},p} \overset{\text{Obs. 10}}{=} c\,\|\partial^\alpha g_\varepsilon - \partial^\alpha g\|_{\tilde{\Omega}_{\frac{r}{2}},p}$$

$$\leq \underbrace{\|\underset{\underset{\rho_\varepsilon * \partial^\alpha g^\varepsilon}{\parallel}}{\partial^\alpha g_\varepsilon} - \partial^\alpha g^\varepsilon\|_p}_{\longrightarrow 0} + \underbrace{\|\partial^\alpha g^\varepsilon - \partial^\alpha g\|_p}_{\longrightarrow 0}$$

by prop's of mollifiers (L^p_{loc} conv.)

by cont. of L^p-transl. (ext. from $\Omega' \subset\subset \Omega$ to $\mathbb{R}^d$)

Hence $f_\varepsilon \to f$ in $W^{m,p}(\Omega_{\frac{r}{2}})$.

4.) Take any $\delta > 0$.

$\partial\Omega$ comp. + 1)–3):

$\exists\, x_{0,1}, \dots, x_{0,N} \in \partial\Omega$, $r_i > 0$, func'ns f_i s.t.

$$\partial\Omega \subset \bigcup_{i=1}^N B(x_{0,i}, \tfrac{r_i}{2})$$

and

$$f_i \in C^\infty(\overline{\Omega_i}), \quad \Omega_i = \Omega \cap B(x_{0,i}, \tfrac{r_i}{2}),$$

$$\|f_i - f\|_{W^{m,p}(\Omega_i)} < \delta.$$

5.) Take open Ω_0 s.t. $\overline{\Omega_0} \subset \Omega$ and $f_0 \in C^\infty(\overline{\Omega_0})$ s.t.

$$\|f_0 - f\|_{W^{m,p}(\Omega_0)} < \delta \qquad \text{(by Thm 6)}$$

5.) Take part. of 1 $\{\varphi_i\}_j$ subord. to $\Omega_0, \dots, \Omega_N$.

Def. $$\tilde{f} = \sum_{i=0}^N f_i \varphi_i$$

Obs: $\tilde{f} \in C^\infty(\overline{\Omega})$ and

$$\|\partial^\alpha \tilde f - \partial^\alpha f\|_{\Omega,p} \underset{\substack{\uparrow \\ f=\sum f\varphi_i}}{\le} \sum_{i=0}^{N} \|\partial^\alpha(f_i\varphi_i) - \partial^\alpha(f\varphi_i)\|_{\Omega_i,p}$$

prod. rule

$$\underset{\text{HW}}{\le} C \sum_{i=0}^{N} \underbrace{\|f_i - f\|_{\Omega_i,|\alpha|,p}}_{\|\varphi_i\|_{\Omega_i,|\alpha|,\infty}} < \tilde C (N+1)\,\delta \quad \text{for } |\alpha| \le m$$

Hence $\|\tilde f - f\|_{\Omega,m,p} \le \sum_{|\alpha|\le m} \|\partial^\alpha \tilde f - \partial^\alpha f\|_{\Omega_i,p} < C_{m,N}\,\delta$ $\square$

3. Intermezzo: Straightening the bnd'ry.

Very usefull tool. to study func'ns up to the bnd'ry.
"Reduce to (loc.) flat bnd'ry"
$\partial\Omega$ C^m; $x_0 \in \partial\Omega$; $r, \gamma \in C^m$, L given by Def. 9:

$$L\Omega \cap B((x_0',0),r) = \{x \in B((x_0',0),r) : x_d > \gamma(x')\}$$

Def. 11: Straightening map

(i) $S : L\Omega \cap LB(x_0, r) \to \mathbb{R}^d_+ = \{(x_1, \dots, x_d) : x_d > 0\}$

$$y = S(x) = (x', x_d - \gamma(x'))$$

(ii) $\Phi = S \circ L : \Omega \cap B(x_0, r) \to \mathbb{R}^d_+$

Obs. 12:

$$x = S^{-1}(y) = (y', y_d + \gamma(y'))$$

Chk.: $\det DS = 1 = \det DS^{-1}$ (Jacobian)

$S, S^{-1}, \Phi, \Phi^{-1} = L^{-1} \circ S^{-1}$ inv., vol. preserving

Lem. 13:

Let $\partial\Omega$ C^m, $\tilde{\Omega} := \Omega \cap B(x_0, r)$, $u \in W^{m,p}(\tilde{\Omega})$, and def.

$$v(y) := u(\Phi^{-1}(y)) \quad \text{for} \quad y \in \Phi(\tilde{\Omega})$$

Then $v \in W^{m,p}(\Phi(\tilde{\Omega}))$ and

(i) $\|v\|_{L^p(\Phi(\tilde{\Omega}))} = \|u\|_{L^p(\tilde{\Omega})}$

(ii) $\|\partial^\alpha v\|_{L^p(\Phi(\tilde{\Omega}))} \le \|\Phi^{-1}\|_{\tilde{\Omega}, |\alpha|, \infty} \|u\|_{\tilde{\Omega}, |\alpha|, p}$

(iii) $\|\partial^\alpha u\|_{L^p(\tilde{\Omega})} \le \|\Phi\|_{\Phi(\tilde{\Omega}), |\alpha|, \infty} \|v\|_{\Phi(\tilde{\Omega}), |\alpha|, p}$

Pf.: HW (do $|\alpha| = 1$ case)

Hint: $\partial_i (u(\Phi^{-1}(x))) = \sum u_{x_j} \cdot \partial_i \Phi^{-1}_j$; $\det D\Phi = 1$ $\square$

[See Adams Thm. 3.41]