Sobolevrom lec. 19 5.4.2019

Part 5: Application

Porous Medium Eq'n (PME)

Plan:

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A. PME

B. Num. approx.

C. Apriori estim.

D. Compactness, conv. subseq'nce

E. Conv. of full seq'nce

A. PME

$$(1)\quad \begin{cases} u_t = \Delta(u^m) & \text{in } \mathbb{R}^d \times (0,T),\ m>0 \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d \qquad (u_0 \geq 0) \end{cases}$$

$m = 1$: Heat eq'n

$m > 1$: Slow diffusion, phys. porous medium, $\exists$ sol'ns w. comp. supp., mass preserved

$0 < m < 1$: Fast diff., never comp. supp.,

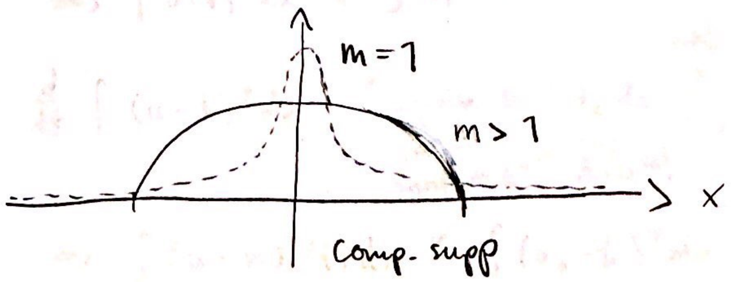
can vanish in finite time ($m < m_c$).

<u>Self-similar sol'ns</u> (fundamental sol'ns):

$m = 1$: $u_F(x,t) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$

$m > 1$: $u_F(x,t) = \max\left\{0,\ t^{-\alpha}\left(1 - \frac{\alpha(m-1)}{2dm} \frac{|x|^2}{t^{2\alpha/d}}\right)^{\frac{1}{m-1}}\right\}$,

$\alpha = (m - 1 + \frac{2}{d})^{-1}$



Non-smooth sol'ns ($m > 1$) !

<u>Modelling ($m > 1$):</u> Density of gas in P.M.

$\rho_t + \nabla\cdot(\rho v) = 0$ cons. of mass

$v = -\nabla p$ Darcy's law

$\rho = p^{\gamma}$ eq'n of state ($\gamma = 1$ ideal gas)

$\Rightarrow \rho_t = \nabla\cdot(\rho \nabla \rho^{\frac{1}{\gamma}}) = \frac{1}{1+\frac{1}{\gamma}} \Delta(\rho^{1+\frac{1}{\gamma}})$

<u>A priori estim. ($m \geq 1$):</u>

(a) $\int u(x,t)\, dt = \int u_0\, dx$

(b) $u_0 \geq 0 \Rightarrow u \geq 0$

(c) $\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty$
$\cancel{\|u_0\|_\infty < \infty \Rightarrow \|u(\cdot,t)\|_\infty < \infty}$

3.)

(d) $\|u(\cdot,t)\|_1 \le \|u_0\|_1$ ~~$\|u_0\|_1 < \infty \Rightarrow \|u(\cdot,t)\|_1 < \infty$~~

(e) $\int \Phi(u(x,t))\,dx + \int_0^t \int |\nabla(u^m)|^2\,dx\,dt = \int \Phi(u_0)\,dx,$ where $\Phi(r) = \int_0^r s^m\,ds.$

(f) $\int (u^1 - u^2)^+(x,t)\,dx \le \int (u_0^1 - u_0^2)^+\,dx$

(g) $u_0^1 \le u_0^2 \Rightarrow u^1 \le u^2$

"Pf.:"

(a) $\frac{d}{dt}\int u = \int u_t = \int \Delta(u^m) = \lim_{R\to\infty} \int_{|x|=R} \nabla(u^m)\cdot n\,dS \overset{\text{if } u \to 0,\ |x|\to\infty}{=} 0$

$\overset{\int_0^t dt}{\Longrightarrow} \int u(t)\,dx = \int u_0\,dx$

(b)–(d) $\frac{d}{dt}\int (u-k)^+\,dx = \int \underbrace{\operatorname{sgn}(u-k)^+}_{\overset{\text{chk}}{=}\ \operatorname{sgn}(u^m-k^m)^+} \underbrace{u_t}_{\overset{(1)}{=}\ \Delta(u^m)}\,dx \underset{(!)}{\overset{\text{chk}}{\le}} \int \Delta(u^m - k^m)^+ = \dots = 0$

(*) $\Rightarrow \int (u-k)^+(t)\,dx \le \int (u_0-k)^+\,dx$

$k=0$: $\int u^+(t) \le \int u_0^+$,

Similar ... $\int u^-(t) \le \int u_0^-$

Then $u_0 \ge 0 \Rightarrow u^- = 0 \Rightarrow u \ge 0$ [(b)]

$k = \|u_0\|_\infty$: $\int (u - \|u_0\|_\infty)^+(t) \le 0 \Rightarrow u \le \|u_0\|_\infty$ [(c)]

$k \longmapsto \int |u(t)| = \int u^+(t) + u^-(t) = \int u_0^+ + u_0^- = \int |u_0|$ [(d)]

(e) $(1)\cdot u^m + \int dx \overset{\text{i.b.p.}}{\dashrightarrow} \frac{d}{dt}\Phi(u) = -\int |\nabla(u^m)|^2\,dx$

(f) Vazquez: PME, Thm. 6.6 p. 134

(g) $u_0^1 \le u_0^2 \overset{(f)}{\Longrightarrow} u^1 \le u^2$

□

Generalized PME (GPME)

$$(2)\begin{cases} u_t = \Delta\varphi(u) & \text{in } \mathbb{R}^d\times(0,T) =: Q_T \\ u(x,0) = u_0 & \text{in } \mathbb{R}^d \end{cases}$$

(A1) $\varphi \in W^{1,\infty}_{loc}(\mathbb{R})$, $\varphi' \geq 0$, $\varphi(0) = 0$

(A2) $u_0 \in L^1 \cap L^\infty$, $u_0 \geq 0$

OBS: $\varphi(r) = r^m$ satisfy (A1) when $m \geq 1$.

Thm. 1: $\exists!$ $\mathcal{D}'$-sol'n $0 \leq u \in L^\infty(Q_T) \cap C_b([0,T]; L^1(\mathbb{R}^d))$ of (2) satisfying (a) – (g).

Rem. 2:

a) $\mathcal{D}'$-sol'n:

$$\iint (u\psi_t + \varphi(u)\Delta\psi)\,dx\,dt + \underbrace{\int (u\psi)(x,0)\,dx}_{\text{init. cond'n}} = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^d\times[0,T))$$

b) $\mathcal{D}'$-sol'n + (e) $\Rightarrow$ weak energy sol'n:

$$\iint (u\psi_t - \underbrace{\nabla\varphi(u)}_{\exists}\cdot\nabla\psi)\,dx\,dt + \int (u\psi)(x,0)\,dx = 0$$
$$\forall \psi \in C_c^1(\mathbb{R}^d\times[0,T))$$

c) u cl. sol'n where it is smooth.

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B. Num. approx.

Monotone explicit FDM

$$(3)\begin{cases} \dfrac{U^{n+1} - U^n}{\Delta t} = \Delta_h \varphi(U^n) = \dfrac{\varphi(U^n)(x+h) - 2\varphi(U^n)(x) + \varphi(U^n)(x-h)}{h^2} \\ U^0 = u_0 \end{cases}$$

where

$$\Delta_h = D_h^+ D_h^- = \frac{\tau_h - 1}{h} \cdot \frac{1 - \tau_{-h}}{h} = \frac{\tau_h - 2 + \tau_{-h}}{h^2}$$

OBS: 1D! ND extension trivial (incl. analysis) !!

Let

$$k := \frac{2\Delta t}{h^2},$$

rewrite (3) (chk):

$$(4) \quad U^{n+1}(x) = \left(U^n - k\varphi(U^n)\right)(x) + \frac{k}{2}\left[\varphi(U^n)(x+h) + \varphi(U^n)(x-h)\right]$$

$$= G\left(U^n(x), U^n(x+h), U^n(x-h)\right)$$

(3) (or (4)) <u>monotone</u> if G non-decreasing:

$$G(\nearrow, \nearrow, \nearrow)$$

Chk.: $\partial_2 G, \partial_3 G \geq 0 \quad (\varphi' \geq 0)$

$$\partial_1 G = (1 - k\varphi'(U^n)) \geq 0$$

$$\text{if} \quad 1 - k\|\varphi'(U^n)\|_\infty \geq 0$$

<u>CFL-cond'n:</u>

$$(\text{CFL}) \quad Kk \leq 1, \qquad K = \sup_{|r| \leq M} |\varphi'(r)|, \quad M = \sup_n \|U^n\|_\infty$$

$$Kk \leq 1 \overset{\text{def } k}{\iff} \Delta t \leq \frac{1}{2K} h^2$$

OBS: $M < \infty \overset{(A1)}{\implies} K < \infty$

C. A priori estim.

Prop. 3: (unif. estim.)

(a) $\|U^n\|_\infty \le \|u_0\|_\infty$

(b) $\|U^n\|_1 \le \|u_0\|_1$

(c) $\|U^n - \tau_\delta U^n\|_1 \le \|u_0 - \tau_\delta u_0\|_1 =: \omega_{u_0}(\delta)$.

$\left(u_0 \in L^1 \overset{\text{cont of } L^1\text{-transl}}{\Longrightarrow} \omega_{u_0}(\delta) \underset{\delta \to 0^+}{\longrightarrow} 0\right.$

(d) $\|U^{n+k} - U^n\|_1 \le \bar{\omega}(k\Delta t)$,

$$\bar{\omega}(r) = \inf_{\varepsilon > 0} \left\{ 2\omega_{u_0}(\varepsilon) + cK \frac{r}{\varepsilon^2} \right\}.$$

[Chk.: $\bar{\omega}(r) \underset{r \to 0}{\longrightarrow} 0$, $\bar{\omega}$ indep. on $x, n, h, \Delta t$]

÷ [(e) $\int |D_h^+ \varphi(U^n)|^2 \le K\|u_0\|_1 + MK\bar{\omega}(\Delta t)$

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Pf.:

(a) $U^{n+1} = G(U^n(x), U^n(x+h), U^n(x-h))$

$\overset{U^n \le V^n}{\le} G(V^n(x), V^n(x+h), V^n(x-h)) = V^{n+1}(x)$

$V^n = \|u_0\|_\infty$
$\Longrightarrow$ (a)

(b) $T[U] = U + \Delta t\, \Delta_h \varphi(U) = G$

$[(4) \iff U^{n+1} = T[U^n]]$

By pf. (a): $U^n \le U^n \vee V^n$

$T[U^n] - T[V^n] \le T[U^n \overset{\max}{\vee} V^n] - T[V^n]$

<u>Lem. 4:</u> u^n, v^n solve (3), then

$$\int (u^n - v^n)^+(x)\, dx \le \int (u_0 - v_0)^+(x)\, dx$$

<u>Pf.:</u>

$$T[u] = u + \Delta t\, \Delta_h \varphi(u) = G[u]$$

$$\left((4) \Leftrightarrow u^{n+1} = T[u^n] \right)$$

$$T[u^n] = G[u^n] \underset{u^n \le u^n \vee v^n}{\le} G[u^n \overset{\max}{\vee} v^n] = T[u^n \vee v^n]$$

$$\Rightarrow T[u^n] - T[v^n] \le T[u^n \vee v^n] - T[v^n]$$

$$0 = T[v^n] - T[v^n] \underset{v^n \le u^n \vee v^n}{\le} T[u^n \vee T^n] - T[v^n]$$

Hence

$$(T[U^n]-T[V^n])^+ \leq T[U^n \vee V^n] - T[V^n]$$

$$\int \text{———} \ \text{"} \ \text{———} dx \leq \int \text{———} \ \text{"} \ \text{———} dx$$

$$\overset{\text{def } T}{=} \int \underbrace{(U^n \vee V^n - V^n)}_{=(U^n-V^n)^+} - \underbrace{\int \Delta t \Delta_h (\varphi(U^n \vee V^n) - \varphi(U^n))\, dx}_{\overset{\text{def } \Delta_h}{=} \int (\dots)(x-h)dx - 2\int(\dots)(x) + \int(\dots)(x+h)}$$

$$= 0$$

$$[\,|\varphi(U)| \leq \underset{0}{|\varphi(0)|} + K|U| \in L^1 \text{ if } U \in L^1]$$

$$\Updownarrow \quad \begin{aligned} U^{n+1} &= T[U^n] \\ V^{n+1} &= T[V^n] \end{aligned}$$

$$\int (U^{n+1}-V^{n+1})^+ \leq \int (U^n - V^n)^+ \leq \dots \leq \int (u_0 - v_0)^+ \quad \square$$

<u>Pf. Prop. 3:</u>

(a) $U^{n+1} = G[U^n] \underset{U^n \leq V^n}{\leq} G[V^n] = V^{n+1}$

$\overset{V^n = \|u_0\|_\infty}{\Longrightarrow} \quad U^n \leq \|u_0\|_\infty$. Similarly, $-\|u_0\|_\infty \leq U^n$

(b) Lem. 4 w. $V^n = 0$ [and then $(u,v) = (0,u)$]

(c) Lem. 4 w. $V^n = \tau_\delta U^n$ [and then $(u,v) = (\tau_\delta u, u)$]

(d) $U^n_\varepsilon = u^n *_x \rho_\varepsilon$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, $0 \leq \rho \in C_c^\infty$, $\int \rho = 1$, $\operatorname{supp} \rho \subset B(0,1)$

$$U^{n+1}_\varepsilon \underset{\text{chk.}}{\overset{(3)}{=}} U^n_\varepsilon + \Delta t\, \varphi(\underset{\approx}{U^n}) * \Delta_h \rho_\varepsilon$$

$\Downarrow$

$$\|u_\varepsilon^{n+1} - u_\varepsilon^n\|_1 \le \Delta t \underbrace{\|\varphi(u^n)\|_{\infty}}_{\le K\|u^n\|_1} \cdot \underbrace{\|\Delta_h \rho_\varepsilon\|_1}_{\le c\|\partial^2\rho_\varepsilon\|_1}$$

chk.

$$\le c\frac{1}{\varepsilon^2}\|\partial^2\rho\|_1$$

$\Downarrow$

$$\|u_\varepsilon^{n+k} - u_\varepsilon^n\|_1 \le \sum_{j=1}^{k} \|u_\varepsilon^{n+j} - u_\varepsilon^{n+j-1}\|_1 \le C\frac{k\Delta t}{\varepsilon^2}$$

$\Downarrow$

$$\|u^{n+k} - u^n\|_1 \le \underbrace{\|u^{n+k} - u_\varepsilon^{n+k}\|_1}_{\le \sup_{|\delta|\le\varepsilon}\|u^{n+k} - \tau_\delta u^{n+k}\|_1 \le \omega_{u_0}(\varepsilon)} + \underbrace{\|u_\varepsilon^{n+k} - u_\varepsilon^n\|_1}_{\le C\frac{k\Delta t}{\varepsilon^2}} + \underbrace{\|u_\varepsilon^n - u^n\|_1}_{\le \omega_{u_0}(\varepsilon)}$$

earlier

$$\le \sup_{|\delta|\le\varepsilon}\|u^{n+k} - \tau_\delta u^{n+k}\|_1$$

$$\le \omega_{u_0}(\varepsilon)$$

$\square$