Sobolevrom lec. 20 8.4.2019

Last time:

(1) $\begin{cases} u_t = \varphi(u)_{xx} & \text{in } \mathbb{R}\times(0,T) =: Q_T \\ u(x,0) = u_0 & \text{in } \mathbb{R} \end{cases}$

(2) $\begin{cases} \dfrac{U^{n+1}-U^n}{\Delta t} = \Delta_h \varphi(U^n) & \text{in } \mathbb{R},\ n = 1,2,\dots,\frac{T}{\Delta t} \\ U^0 = u_0 & \text{in } \mathbb{R} \end{cases}$

(A1) $\varphi \in W^{1,\infty}_{loc}$, $\varphi' \geq 0$ (a.e.), $\varphi(0) = 0$

(A2) $u_0 \in L^\infty \cap L^1$, $K = \sup_{|r|\leq M} |\varphi'(r)|$, $M = \sup_n \|U^n\|_\infty \ (= \|u_0\|_\infty)$

(CFL) $\Delta t \leq \frac{h^2}{2K}$

Prop. 3: (A priori estim.) CFL - cond !

(a) $\|U^n\|_\infty \leq \|u_0\|_\infty$

(b) $\|U^n\|_1 \leq \|u_0\|_1$

(c) $\|U^n - \tau_\delta U^n\|_1 \leq \|u_0 - \tau_\delta u_0\|_1 =: \omega_{u_0}(\delta)$

(d) $\|U^{n+k} - U^n\|_1 \leq \bar{\omega}(k\Delta t)$, $\bar{\omega}$ only dep. on u_0, φ

D. Compactness

i) <u>Interpolation in t:</u>

$$\tilde{u}(x,t) = \frac{(n+1)\Delta t - t}{\Delta t} U^n(x) + \frac{t - n\Delta t}{\Delta t} U^{n+1}(x), \quad t \in [n\Delta t, (n+1)\Delta t)$$

p. wise lin. interp.

Prop 3P $\Longrightarrow$ $\tilde{u} \in L^\infty(Q_T) \cap C_b([0,T]; L^1(\mathbb{R}))$ and

(a')/(b') $\quad \|\tilde{u}(\cdot,t)\|_{\infty/1} \le \|u_0\|_{\infty/1}$

(c') $\quad \|\tilde{u}(\cdot,t) - \tau_\delta \tilde{u}(\cdot,t)\|_1 \le \omega_{u_0}(\delta)$

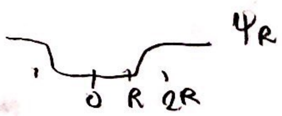
(d') $\quad \|\tilde{u}(\cdot,t) - \tilde{u}(\cdot,s)\|_1 \le 3\bar{\omega}(|t-s| + \Delta t)$

$$\Big[\int |\tilde{u}(t) - \tilde{u}(s)| \underset{\text{choose } k,m\ldots}{\le} \underbrace{\int |\tilde{u}(t) - \tilde{u}(k\Delta t)|}_{\le \bar{\omega}(\Delta t)} + \underbrace{\int |\tilde{u}(k\Delta t) - \tilde{u}(m\Delta t)|}_{\substack{\le \bar{\omega}((k-m)\Delta t) \\ \le \bar{\omega}(|t-s|)}} + \underbrace{\int |\tilde{u}(m\Delta t) - \tilde{u}(s)|}_{\le \bar{\omega}(\Delta t)} \Big]$$

OBS: $\tilde{u} = \tilde{u}_{h,\Delta t}$

ii) <u>Tightness:</u>

$\psi_R(x) = 1 - \varphi\left(\frac{x}{R}\right)$, $\varphi \in C_c^\infty$, $0 \le \varphi \le 1$, $= 1$ in $[-1,1]$, $= 0$ in $[-2,2]^c$



$$\int U^{n+1}\psi_R \overset{(2)}{=} \int U^n \psi_R + \Delta t \underbrace{\int \Delta_h \varphi(U^n)\psi_R}_{\underset{\text{chk}}{=} \int \varphi(U^n) \underbrace{\Delta_h \psi_R}_{\sim \partial^2 \psi_R \sim \frac{1}{R^2}}}$$

$$= \int U^n \psi_R - O\left(\frac{\Delta t}{R^2} \|\varphi(U^n)\|_1\right)$$

$$\vdots$$

$$= \int u_0 \psi_R - O\left(\frac{T}{R^2}\right), \quad (n\Delta t \le T)$$

Hence

$$\sup_{n,h,\Delta t} \int_{|x|>2R} |U^{n+1}| \overset{U^{n+1}\geq 0}{\leq} \sup_{n,h,\Delta t} \int U^{n+1} \psi_R$$

$$\leq \int u_0 \psi_R + K \sup_{n,h,\Delta t} \|U^n\|_1 \frac{T}{R^2}$$

$$\leq \int_{|x|>R} u_0 + C\frac{T}{R^2} \xrightarrow[R\to\infty]{} 0,$$

and then

<u>Lem. 5:</u> $\sup_{h,\Delta t} \int_{|x|>R} |\tilde{u}(\cdot,t)| \xrightarrow[R\to 0]{} 0 \quad \forall t\in[0,T]$

iii) <u>$\{\tilde{u}(\cdot,t)\}_{h,\Delta t>0}$ precomp. in L^1 $\forall t\in[0,T]$:</u>

Kolmogorov comp. thm + $\underset{\text{bnd in } L^1}{(b')}$, $\underset{\text{equicont in } L^1}{(c')}$, $\underset{\text{tight}}{\text{Lem 5}}$

iv) <u>$\{\tilde{u}\}_{h,\Delta t>0}$ precomp. in $X = C([0,T], L^1(\mathbb{R}))$:</u>

Arzela-Ascoli in X + $\underset{\text{pt.w. precomp. in } L^1}{\text{iii)}}$ and $\underset{\text{equicont.}}{(d')}$

Conclusion: If $h_n, \Delta t_n \xrightarrow[n\to\infty]{} 0$ s.t. (CFL) holds

and $\tilde{u}_n := \tilde{u}_{h_n,\Delta t_n}$, then

<u>Prop. 6:</u> $\exists n_k \to \infty$, $u \in C([0,T]; L^1) \cap L^\infty(Q_T)$

s.t. $\tilde{u}_{n_k} \to u$ in $C([0,T]; L^1(\mathbb{R}))$.

<u>Pf.:</u>

iii) + A.A. $u\in L^\infty$ since $\|u\|_\infty \leq \|u_0\|_\infty$ by (a') $\square$

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E. Convergence

i) Limit eq'n: u D'-sol'n of (1)

$$(2)\cdot \psi(x,t_n) \quad + \int dx \quad + \sum_{n=0}^{N-1} \Delta t$$

$$\Downarrow$$

$$\underbrace{\sum \int \frac{\tilde{u}(x,t_{n+1}) - \tilde{u}(x,t_n)}{\Delta t} \psi(t_n,x)\,dx\,\Delta t}_{\displaystyle \int\Big[\tilde{u}(t_N)\psi(t_{N-1}) - \tilde{u}(0)\psi(0) - \sum_{n=1}^{N-1} \tilde{u}(t_n)\frac{\psi(t_n)-\psi(t_{n-1})}{\Delta t}\Delta t\Big]} = \underbrace{\sum \int \Delta_h \varphi(\tilde{u}(t_n))\psi(t_n)\,dx\,\Delta t}_{= \int \varphi(\tilde{u})\Delta_h \psi\, dx}$$

$\tilde{u}(t_N)\psi(t_{N-1}) \xrightarrow{\text{supp}\,\psi} 0$

$\Delta t \to 0$:

$\psi \in C_c^\infty(\mathbb{R}\times[0,T))$
$\Downarrow$
$\Delta_h \psi \longrightarrow \Delta\psi$ in L^∞

$\tilde{u} \longrightarrow u$ in $C([0,T];L^1)$ + $\tilde{u}$ equi-L^∞-bnd
$\Downarrow$
$\varphi(\tilde{u}) \longrightarrow \varphi(u)$ in $C([0,T];L^1)$

$$\iint (u\psi_t + \varphi(u)\Delta\psi) + \int (u\psi)(0,x)\,dx = 0$$

Lem. 7: $\tilde{u} \longrightarrow u$ in $C([0,T];L^1)$ + $\tilde{u}$ solves (2) + $\tilde{u}$ equi-L^∞-bnd.
$\Rightarrow$ u D' sol'n of (1)

ii) Conv. of $\{\tilde{u}_n\}$

i) + uniqueness for (1)

$$\Downarrow$$

Any $C([0,T];L^1)$ conv. subseq'nce of $\{\tilde{u}_n\}_n$ conv. to the same u

$$\begin{array}{c}\Downarrow \qquad \qquad X \\ \qquad \qquad \qquad \| \end{array}$$

$\tilde{u}_n \to u$ in $C([0,T]; L^1)$ (the whole seq'nce)

[If not, $\exists\, \delta > 0$, $\tilde{u}_{n_k}$ s.t. $d_X(\tilde{u}_{n_k}, u) > \delta$

But Prop. 6 + Lem. 7 $\Rightarrow \exists\, \tilde{u}_{n_{k_l}} \to u$

Contradiction]

Conclusion:

Thm. 8: $\exists\, \mathcal{D}'$-sol'n $u \in C([0,T]; L^1(\mathbb{R})) \cap L^\infty(Q_T)$

of (1) and

$$\tilde{u}_n \longrightarrow u \quad \text{in} \quad C([0,T], L^1(\mathbb{R}))$$

Pf.:

$\exists$ + conv. : done above

$u \in L^\infty(Q_T)$: Prop 3 (a) + $u_{n_k} \xrightarrow{a.e.} u$

$$\Rightarrow \|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty \qquad \square$$

Rem. 5:

a) $\exists$ of $\mathcal{D}'$-sol'n of (1)

b) Conv. of num. meth. (2)

c) u inherits a priori estim's from Prop 3:

(a) $\|u(t)\|_\infty \leq \|u_0\|_\infty$ (c) $\|\tau_\delta u(t) - u(t)\|_1 \leq \omega_{u_0}(\delta)$

(b) $\|u(t)\|_1 \leq \|u_0\|_1$ (d) $\|u(t+\delta) - u(t)\|_1 \leq 3\bar{\omega}(\delta)$

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