Sobolevrom lec. 19 Mon 25-3-19

Sobolev ineq.:

$1 \le p < d$: $\|f\|_{L^{p^*}} \overset{\Omega \text{ bnd.}}{\lesssim} C\|f\|_{W^{1,p}}$, $p^* = \frac{pd}{d-p}$ (Ga. Ni. So.)

$d < p \le \infty$: $\|f\|_{C^{0,\gamma}} \overset{\Omega \text{ bnd.}}{\lesssim} C\|f\|_{W^{1,p}}$, $\gamma = 1 - \frac{d}{p}$ (Morrey)

11. General Sobolev ineq'ies

Thm. 35: $\Omega \subset \mathbb{R}^d$, bnd., open, $\partial\Omega$ C^1, $k \in \mathbb{N}_0$, $p \ge 1$.

Assume $f \in W^{k,p}(\Omega)$.

(i) If $k < \frac{d}{p}$, then $f \in L^q(\Omega)$ for

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d},$$

and $\exists C = C_{k,p,d,\Omega}$ s.t.

$$\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{k,p}(\Omega)}.$$

(ii) If $k > \frac{d}{p}$, then $f \in C^{k-[\frac{d}{p}]-1,\gamma}(\bar{\Omega})$ for

$$\gamma = \begin{cases} [\frac{d}{p}] + 1 - \frac{d}{p}, & \frac{d}{p} \notin \mathbb{N} \\ \text{any } \gamma \in (0,1), & \frac{d}{p} \in \mathbb{N} \end{cases}$$

and $\exists C = C_{k,p,d,\gamma,\Omega}$ s.t.

$$\|f\|_{C^{k-[\frac{d}{p}]-1,\gamma}(\bar{\Omega})} \le C\|f\|_{W^{k,p}(\Omega)}.$$

<u>Rem. 36:</u> $\forall\, m, p \;\exists\, k$ s.t. $W^{k,p} \subset C^m$.

$$W^{\infty,p} \subset C^\infty \quad \forall\, p$$

<u>Pf.:</u>

(i) <u>$k < \frac{d}{p}$</u>; $\partial^\alpha f \in L^p \;\forall\, |\alpha| \le k$

$\Rightarrow \partial^\beta f \in W^{1,p}$, $|\beta| \le k-1$.

Obs: $p < \frac{d}{k} < d$.

$$\overset{\text{Ga.Na.So.}}{\underset{\text{Thm. 24}}{\Longrightarrow}} \|\partial^\beta f\|_{L^{p^*}} \le C \|\partial^\beta f\|_{W^{1,p}} \le C\|f\|_{W^{k,p}}, \; |\beta| \le k-1$$

$\Longrightarrow f \in W^{k-1,p^*}$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$

...

$\overset{\partial^{k-2} f \in W^{1,p^*}}{\Longrightarrow} f \in W^{k-2,p^{**}}$, $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{d} = \frac{1}{p} - \frac{2}{d}$

$\vdots$

$\Longrightarrow f \in W^{0,q} = L^q$, $\frac{1}{q} = \underbrace{\frac{1}{p} - \frac{k}{d}}$ (> 0 since $k < \frac{d}{p}$)

Hence: $\|f\|_{L^q} \le \dots \le \tilde{C}\|f\|_{k-1,p^*} \le C\|f\|_{k,p}$

(ii) <u>$\frac{d}{p} \notin \mathbb{N}$</u>: $\left(k > \frac{d}{p}\right)$

$f \in W^{k,p} \Rightarrow \partial^\alpha f \in W^{l,p}$, $|\alpha| \le k-l$, hence by (i),

(*) $f \in W^{k-l,r}$ for all l, r s.t. $\frac{1}{r} = \frac{1}{p} - \frac{l}{d}$ and $l \cdot p < d$

Take $l = \left[\frac{d}{p}\right]$, then $l < \frac{d}{p} < l+1$ and hence

$l \cdot p < d$ and $r = \dfrac{pd}{d - pl} \underset{\frac{d}{p} < l+1}{>} d$.

By (*) and Morrey (Thm. 33),

$$\partial^\alpha f \in W^{1,r} \subset C^{0,1-\frac{d}{r}} \quad \forall\ |\alpha| \le k - l - 1$$

$$\Downarrow \quad 1 - \frac{d}{r} \overset{\text{def. } r}{=} 1 - \frac{d}{p} + l \quad [\tfrac{d}{p}]$$

$$f \in C^{k - [\frac{d}{p}] - 1,\ [\frac{d}{p}] + 1 - \frac{d}{p}}(\overline{\Omega})$$

<u>$\frac{d}{p} \in \mathbb{N}$:</u> Let $l = [\frac{d}{p}] - 1 = \frac{d}{p} - 1$. By (*)

$$f \in W^{k-l,r} \quad \text{for} \quad r = \frac{pd}{d - pl} \overset{\text{def. } l}{=} d \qquad (l \cdot p < d)$$

$\Downarrow$ Ga.Ni.So. (Thm. 24)

$$\partial f^\alpha \in L^q \ (\forall)\ d \le q < \infty,\ |\alpha| \le k - l - 1 = k - [\tfrac{d}{p}]$$

$\Downarrow$ Morrey (Thm. 33)

$$\partial f^\alpha \in \overset{W^{1,p} \subset}{C^{0,1-\frac{d}{q}}} \quad \forall\ d < q < \infty,\ |\alpha| \le k - [\tfrac{d}{p}] - 1$$

$\Downarrow$

$$f \in C^{k - [\frac{d}{k}] - 1, \gamma} \quad \forall\ 0 < \gamma < 1 \qquad \square$$

12. Embeddings in Banach sp's

X, Y Banach sp's

$\Phi : X \to Y$

<u>Def. 37:</u>

(a) Φ (Banach sp.) embedding if lin., cont., 1-1

(b) Φ cont. embedding, $\Phi : X \hookrightarrow Y$, if

$$\exists C > 0 \ \text{s.t.} \ \|\Phi x\|_Y \le C \|x\|_X \quad \forall x \in X$$

(c) Φ comp. emb., $\Phi : X \hookrightarrow\hookrightarrow Y$, if

(i) $\Phi : X \hookrightarrow Y$

(ii) Φ comp., i.e. $\Phi(B)$ precomp. $\forall B \subset X$ bnd.

($\Leftrightarrow \Phi(B)$ seq'al – " —)

(d) $X \subset Y$ cont. emb. in Y if $id : X \hookrightarrow Y$.

(e) $X \subset Y$ comp. emb. in Y if $id : X \hookrightarrow\hookrightarrow Y$.

[$\|x\|_Y \le C\|x\|_X$, $\{x_n\} \subset X$ bnd. $\Rightarrow \exists \{x_{n_k}\}_k$ Cauchy in Y]

Ex. 38: $\Omega \subset \mathbb{R}^d$ open, bnd.; $\partial\Omega$ C^1

$id : W^{1,p}(\Omega) \overset{1 \le p < d}{\hookrightarrow} L^{p^*}(\Omega)$ (Ga.-Ni.-So.)

$id : W^{1,p}(\Omega) \overset{p > d}{\hookrightarrow} C^{0,1-\frac{d}{p}}(\bar{\Omega})$ (Morrey)

$id : W^{1,p}(\Omega) \overset{1 \le p < d}{\hookrightarrow\hookrightarrow} L^q(\Omega)$, $1 \le q < p^*$ (Rellich, today)

15:02

Lem. 39: $X \subset Y \subset Z$ normed sp's, $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$.

Then $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ s.t.

$$\|x\|_Y \le \varepsilon\|x\|_X + C_\varepsilon \cdot \|x\|_Z \quad \forall x \in X.$$

Pf.: HW, see Holden p. 19.

Ex. 40: Ω bnd., $p < d$

$W^{2,p} \overset{\text{Ex. 38}}{\hookrightarrow\hookrightarrow} W^{1,p} \overset{\text{Ex. 38}}{\hookrightarrow\hookrightarrow} L^p \overset{\text{Lem 39}}{\Rightarrow} \|\nabla f\|_p \le \varepsilon\|\partial^2 f\|_p + C_\varepsilon\|f\|_p$

13. Str. comp. in $W^{1,p}(\Omega)$

Bnd. subsets of $W^{1,p}$ are precomp. in other sp's.

Thm. 41: (Rellich–Kondrachov comp. thm.)

$\Omega \subset \mathbb{R}^d$, bnd., open, $\partial\Omega$ C^1.

(a) $1 \le p < d$: $W^{1,p}(\Omega) \underset{\text{comp. emb.}}{\hookrightarrow\hookrightarrow} L^q \quad \forall\, q \in [1, p^*)$

(b) $p = d$: $W^{1,d}(\Omega) \hookrightarrow\hookrightarrow L^q \quad \forall\, q \in [1, \infty)$

(c) $p > d$: $W^{1,d}(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma}(\bar{\Omega}) \quad \forall\, \gamma \in (0, 1 - \frac{d}{p})$

Obs. 42:

$$\sup_n \|f_n\|_{W^{1,p}(\Omega)} < \infty \implies \exists\, f_{n_k}, \tilde{f} \text{ s.t.}$$

$$f_{n_k} \to \tilde{f} \text{ in } \begin{cases} L^q, & 1 \le p < d \\ C^{0,\gamma}, & p > d \end{cases}$$

"Arzela–Ascoli ⇝ Kolmogorov–Riesz ⇝ Rellich–Kondrachov"
$\quad C^0 \qquad\qquad\qquad L^p \qquad\qquad\qquad W^{1,p}$

Pf.:

(b) Follows from (a) since $W^{1,d} \underset{p<d}{\hookrightarrow} W^{1,p}$ and $p^* \xrightarrow[p \to d^-]{} \infty$.

(c) By Morrey (Thm. 33), $W^{1,p} \hookrightarrow C^{0,1-\frac{d}{p}}$.

By Arzela–Ascoli (see Part 1), $C^{0,\gamma} \underset{\gamma > \gamma' > 0}{\hookrightarrow\hookrightarrow} C^{0,\gamma'}$ [HW]

and (c) follows.

(a) $(1 \leq p < d)$

1) <u>Cont. emb.:</u> $W^{1,p} \hookrightarrow L^{p^*} \hookrightarrow L^q, \quad q \in [1, q^*)$

$\uparrow$ Ga.Ni.So Thm. 24 $\quad \nwarrow$ Ω bnd

2) <u>Comp. in L^p:</u> $W^{1,p} \hookrightarrow\hookrightarrow L^p$

$\{f_n\}_n \subset W^{1,p}(\Omega)$ bnd.

$\Downarrow$ extension (Thm. 15)

$\{Ef_n\}_n \subset W^{1,p}(\mathbb{R}^d)$ bnd., $\operatorname{supp} Ef_n \subset B$ bnd.
$(\Omega \subset B)$

$\Downarrow$

i) $\sup_n \|Ef_n\|_p < \infty$

ii) $\sup_n \|Ef_n - \tau_h Ef_n\|_p^p = \sup_n \int |\underbrace{Ef_n(x) - Ef_n(x+h)}_{\leq |h| \int_0^1 |\nabla Ef_n(x+th)|\, dt}|^p\, dx$

$$\overset{\text{Jensen}}{\leq} |h|^p \sup_n \int \int_0^1 |\nabla Ef_n(x+th)|^p\, dt\, dx$$

$$\overset{\text{Fubini}}{=} |h|^p \sup_n \|\nabla Ef_n\|_p^p \overset{\text{ext.}}{\leq} C|h|^p \underbrace{\sup_n \cancel{\|\nabla f_n\|_p^p}}_{< \infty} \quad \|f_n\|_{W^{1,p}(\Omega)}^p$$

iii) $\sup_n \int_{\mathbb{R}^d \setminus B} |Ef_n|^p = 0$

$\Downarrow$ Kolmogorov comp. thm. (Thm , part 3)

$\{Ef_n\}_n$ precomp. in $L^p(\mathbb{R}^d)$

3) <u>Comp. in L^q:</u> $W^{1,p} \hookrightarrow\hookrightarrow L^q, \quad q \in [1, p^*)$

$$\sup_n \|Ef_n\|_{p^*} \underset{\text{Thm. 24}}{\overset{\text{Ga.Ni.So.}}{\leq}} C \sup_n \|Ef_n\|_{1,p} \overset{\text{ext.}}{\leq} \tilde{C} \underbrace{\sup_n \|f_n\|_{1,p}}_{< \infty} =: K$$

By 2), $\exists\, Ef_{n_k}$ Cauchy in L^p, and by interpolation (part 3):

$$\|Ef_{n_k} - Ef_{n_l}\|_q \le \|Ef_{n_k} - Ef_{n_l}\|_1^{\theta} \cdot \underbrace{\|Ef_{n_k} - Ef_{n_l}\|_{p^*}^{1-\theta}}_{\le (2K)^{1-\theta}}$$

$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$ for some $\theta \in [0,1]$

$$\le (2K)^{1-\theta} C \|Ef_{n_k} - Ef_{n_l}\|_p^{\theta}$$

Ω bnd.
$p \ge 1$

$\Rightarrow \{Ef_{n_k}\}_k$ Cauchy in $L^q(\mathbb{R}^d)$ $\forall q \in [1, p^*)$

$\Rightarrow \{f_{n_k}\}_k$ Cauchy/precomp. in $L^q(\Omega)$, —"— $\square$