15. W. conv. and comp. in L^1

L^1 not dual of any Banach sp.

$\Rightarrow$ can not def. w.$*$ conv. in L^1

L^1 not reflexive

$\Rightarrow$ Eberlein-Smuljan does not apply

Ex. 43: Concentration

$$f_n = n^{\frac{1}{p}} \chi_{(0,\frac{1}{n})}$$

$$\|f_n\|_p = 1, \quad p \in [1,\infty)$$

$$\int f_n \varphi = n^{\overbrace{\frac{1}{p}-1}^{\leq 0}} \left(n \int_0^{\frac{1}{n}} \varphi(x)\,dx \right) \longrightarrow \begin{cases} 1 \cdot \varphi(0), & p = 1 \\ 0 \cdot \varphi(0), & p > 1 \end{cases}$$

$$\Rightarrow \begin{cases} f_n \xrightarrow{\mathcal{D}'} \delta_0 \notin L^1, \; p = 1 & \text{(No limit in } L^1\text{!)} \\ f_n \xrightarrow{\mathcal{D}'} 0 \in L^p \; (\overset{E.S.}{\Longrightarrow} f_{n_k} \rightharpoonup 0 \text{ in } L^p, \; p > 1) \end{cases}$$

$[f_n \to 0$ a.e. and in meas. $\forall\, p \in [1,\infty)]$

Ex. 44: Mass escaping to ∞

$f_n(x) = \chi_{(n+1,n)}(x)$, $\|f_n\|_p = 1$, $p \in [1,\infty)$, $f_n \xrightarrow{\mathcal{D}'} 0$, $f_n \rightharpoonup 0$ in L^p, $p \in (1,\infty)$, but $f_n \not\rightharpoonup 0$ in L^1

$[\int f_n \varphi \overset{\varphi=1}{=} \|f_n\|_1 = 1]$

Rem. 45: Bnd'ness $\not\Rightarrow$ w. seq'al comp. in L^1!

~~Need also equi-int.~~

BUT: Bnd'ness in L^1

$\Rightarrow$ seq'al comp. in $\mathcal{M}$, sp. of signed Radon meas.

(We identify L^1 w. subset of $\mathcal{M} = (C_0)'$: $f \mapsto \int f\varphi$, $\varphi \in C_0$

Obs: $\delta_0 \in \mathcal{M}$)

W. seq'al comp. in L^1 need also equi int.

$\{f_n\}_n \subset L^1(X)$ is <u>equiintegrable</u> if:

(i) $\forall \varepsilon > 0 \ \exists B \subset X$ meas., $\mu(B) < \infty$ s.t.

$$\sup_n \int_{X \setminus B} |f_n| \, d\mu < \varepsilon \quad \text{(tightness!)}$$

(ii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$A \subset X, \ \mu(A) < \delta \implies \sup_n \int_A |f_n| \, d\mu < \varepsilon$$

Obs: (ii) $\Rightarrow$ no consentrations; (i) $\Rightarrow$ no mass escaping to ∞!

<u>Thm. 46:</u> (Dunford – Pettis) $\{f_n\}_n \subset L^1(X)$

$\{f_n\}$ w. seq'ally precomp. in L^1

$$\Updownarrow$$

$\{f_n\}_n$ bnd. ($\sup_n \|f_n\|_1 < \infty$) and equiint.

Pause 4:59

<u>Pf. $\Uparrow$):</u>

1) We may assume $f_n \geq 0$

[$f_n = f_n^+ - f_n^-$, $\{f_n^+\}_n$ and $\{f_n^-\}_n$ also bnd + equiint.
W. seq'al comp. of $\{f_n^+\}_n$ and $\{f_n^-\}_n \Rightarrow$ w. seq'al comp. of $\{f_n\}_n$]

2) We may assume $\mu(X) < \infty$ by (i).

[W. seq'al comp. for $\mu(X) < \infty \Rightarrow$ gen. res.:

By (i) let B_k be s.t. $\mu(B_k) < \infty$, $\sup_n \int_{X \setminus B_k} |f_n| < \frac{1}{k}$

Then $\exists$ w. conv. subseq'nce $\{f_{n_k}|_{B_k}\}_{n_k}$ and limit $\tilde{f}_k$ in $L^1(B_k)$

Diag. seq'nce $\{f_{n_n}\}_{n_n}$ conv. w. $\overbrace{\quad}^{\text{to a func'n } \tilde{f}}$ in $L^1(X)$, where $\tilde{f}|_{B_k} = f_k$

For $\varepsilon > 0$; $\varphi \in L^\infty(X)$,

$$\int_X (f_{n_n} - \tilde{f})\varphi = \underbrace{\int_{B_k} (f_{n_n} - \tilde{f})\varphi}_{\substack{\longrightarrow 0 \\ n_n \to \infty}} + \underbrace{\int_{X \setminus B_k} (f_{n_n} - \tilde{f})\varphi}_{\leq \|\varphi\|_\infty \underbrace{\int_{X\setminus B_k} (|f_{n_n}| + |\tilde{f}|)}_{< \frac{1}{k}}} \underset{n \to \infty}{\overset{k,n \text{ large}}{<}} \varepsilon + \varepsilon$$

[chk.: $\tilde{f}$ well-def. + $\tilde{f} \in L^1(X)$ [$L^1(B_k)$ + tight]]

3) Enough to chk. conv. against char. func'ns:

$$\int_X f_n \varphi \to \int_X f\varphi \ \forall \varphi \in L^\infty \iff \int_A f_n \to \int_A f \ \forall A \subset X \text{ meas.}$$

[simple func'ns are dense in $L^\infty(X)$]

4) Conv. subseq'nce f_{n_k} in $\mathcal{M}(X)$:

By bnd'ness and $\mu(X) < \infty$,

$$\sup_n \sup_{\substack{\varphi \in C_0(X) \\ \|\varphi\|_\infty \leq 1}} \left|\int f_n \varphi\right| \overset{\text{Hölder}}{\leq} 1 \cdot \sup_n \int |f_n| \overset{\{f_n\} \text{ bnd.}}{<} \infty.$$

Hence $\{f_n \, d\mu\}$ unif. bnd. in $\mathcal{M}(X) = (C_0(X))'$.

By seq'al comp. in $\mathcal{M}(X)$ (later!), $\exists f_{n_k}$ and $\nu \in \mathcal{M}(X)$ s.t.

$$\int f_{n_k} \varphi \, d\mu \longrightarrow \int \varphi \, d\nu \quad \forall \varphi \in C_0(X)$$

5) $\int_A f_{n_k} d\mu$ Cauchy in $\mathbb{R}$ $\forall A \subset X$ meas.:

Fix $\varepsilon > 0$, $A \subset X$ meas.

Take $C_0 \ni \varphi_k \to \chi_A$ a.e. and $\|\varphi_k\|_\infty \le 1$

[e.g. $\varphi_k = \chi_A * \rho_k$]

By Egorov's thm. $\exists X_\delta \subset X$ meas. s.t.

$$\mu(X \setminus X_\delta) < \delta \quad \text{and} \quad \|\varphi_k - \chi_A\|_{L^\infty(X_\delta)} \to 0$$

Hence

$$\left|\int_A (f_{n_j} - f_{n_l})\right|$$

$$\le \underbrace{\left|\int_X (f_{n_j} - f_{n_l})(\chi_A - \varphi_k)\right|}_{I} + \underbrace{\left|\int_X (f_{n_j} - f_{n_l})\varphi_k\right|}_{\overset{4)}{<} \varepsilon,\ n_j, n_l \text{ big}}$$

$$I \le \underbrace{\left|\int_{X_\delta} -\right|}_{J} + \underbrace{\left|\int_{X \setminus X_\delta} -\right|}_{\le (\underset{1}{\|\chi_A\|_\infty} + \underset{1}{\|\varphi_k\|_\infty})\, 2 \sup_n \int_{X\setminus X_\delta} |f_n| \ \overset{(ii)}{<} \ \varepsilon}$$

δ small (indep. k, n)

$$J \le 2 \sup_n \|f_n\|_1 \|\chi_A - \varphi_k\|_{L^\infty(X_\delta)} < \varepsilon, \quad k \text{ big (indep. } n)$$

and

n_j, n_k big enough $\Rightarrow \int_A (f_{n_j} - f_{n_k}) < 3\varepsilon$.

6) Building a limit:

$$\nu_k(A) := \int_A f_{n_k}\, d\mu$$

$$\tilde{\nu}(A) := \lim_k \nu_k(A) \qquad \forall A \subset X \text{ meas.}$$

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$\tilde{\nu}$ well-def. by 5). Chk: ν_k, $\tilde{\nu}$ pos., finite meas. on X.

$\div$ [≥ 0 by 1), $\nu(X) \leq \sup_n \|f_n\|_1 < \infty$,

$$\nu(A \cup B) = \lim \nu_k(A \cup B) \overset{A \cap B = \emptyset}{=} \lim(\nu_k(A) + \nu_k(B)) = \nu(A) + \nu(B)$$ no uses $\mathcal{V}$ (1)

$A := \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \emptyset$, $\forall \varepsilon > 0\ \exists N$ s.t. for $n \geq N$, $\mu(\bigcup_{i=n}^{\infty} A_i) < \varepsilon$, $\sum_{i=n}^{\infty} \mu(A_i) < \varepsilon$

$$\nu(\bigcup_i A_i) = \sum_{i=1}^{N-1} \nu(A_i) + \nu(\bigcup_{i=N}^{\infty} A_i) \quad (A_j \cap \bigcup_{i=N}^{\infty} A_i = \emptyset \quad j < N)$$

$$= \sum_{i=1}^{\infty} \nu(A_i) \underbrace{- \sum_{i=N}^{\infty} \nu(A_i) + \nu(\bigcup_{i=N}^{\infty} A_i)}_{O(\varepsilon)} \quad]$$

$\tilde{\nu}$ abs. cont. w.r.t. μ:

$$\mu(B) = 0 \implies 0 \leq \tilde{\nu}(B) = \lim_k \int_B f_{n_k}\, d\mu$$

$$\leq \sup_n \int_B |f_n|\, d\mu$$

$$\overset{(ii)}{\underset{\text{equi. int.}}{<}} \varepsilon \quad \forall \varepsilon > 0 \quad (\implies = 0)$$

By Radon-Nikodym, $\exists f \in L^1(X, \mu)$ s.t.

$$\tilde{\nu} = f \cdot \mu$$

7.) Conclusion:

$$\int_A f_{n_k}\, d\mu = \nu_k(A) \to \tilde{\nu}(A) = \int_A f\, d\mu \quad \forall A \subset X \text{ meas.}$$

hence by 3),

$$f_{n_k} \rightharpoonup f \text{ in } L^1 \qquad \square$$

Sobolevrom lec. 15 1.3.2017

Next week:
No lec. Fri
Extra lec. Thurs.

Last time:

Equiint.: (i) $\forall \varepsilon>0\ \exists B\subset X$ meas., $\mu(B)<\infty$ s.t.

$$\sup_n \int_{X\setminus B} |f_n|\, d\mu < \varepsilon$$

(ii) $\forall \varepsilon>0\ \exists \delta>0$ s.t.

$$A\subset X,\ \mu(A)<\delta \Rightarrow \sup_n \int_A |f_n|\, d\mu<\varepsilon$$

Dunford-Pettis: w. seq'al comp. in $L^1 \Leftrightarrow$ bnd. in L^1 + equiint.

res. 47

15. Equiint.

Also called uniform int.

Domination $\underset{(\not\Leftarrow)}{\Rightarrow}$ equiint. (incl. tightness):

$$|f_n| \overset{a.e.}{\le} g\in L^1 \Rightarrow \sup_n \int_A |f_n| \le \int_A g \underset{\uparrow}{\le} \int_A \underset{\overset{\psi}{C_c}}{\tilde g} + \frac{\varepsilon}{2} \le \|\tilde g\|_{L^\infty}\mu(A)+\frac{\varepsilon}{2}<\varepsilon$$

$\|g-\tilde g\|_{L^1}<\frac{\varepsilon}{2}$

$\forall A\subset X, \mu(A)< \frac{\varepsilon}{2\|\tilde g\|_\infty}$

$\sup_n \|f_n\|_p<\infty$, for some $p>1$ + tightness (i) $\Rightarrow$ equiint.:

$$\sup_n \int_A |f_n|\, d\mu \overset{\text{Hölder}}{\le} \underset{\overset{\shortparallel}{\mu(A)^{\frac{1}{q}}}}{\|\chi_A\|_q} \underbrace{\sup_n \|f_n\|_p}_{=:M<\infty} < \varepsilon \text{ when } \mu(A)<\left(\frac{\varepsilon}{M}\right)^q$$

Generally:

Thm. 47: $\{f_n\}_n \subset L^1(X,\mu)$ bnd., $\mu(X) < \infty$

Following statements are equivalent:

(i) $\{f_n\}_n$ equiint.

(ii) $\exists\ \Phi \geq 0$, incr., convex, $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$, s.t.

$$\sup_n \int \Phi(|f_n|)\, d\mu < \infty.$$

(iii) $\lim_{R\to\infty} \sup_n \int_{|f_n|>R} |f_n|\, d\mu = 0$

Rem. 48:

a) $\mu(X) = \infty$: Need in addition tightness!

b) (i) $\Leftrightarrow$ (ii) : de la Vallee Poissin's thm. (Ex.: $\Phi(t) = t^p, p>1$.)

Pf (iii) $\Leftrightarrow$ (i).

$\Rightarrow$) $\sup_n \int_A |f_n|\, d\mu = \sup_n \left(\int_{A\cap\{|f_n|<R\}} + \int_{A\cap\{|f_n|>R\}}\right)$

$$\leq \underbrace{R\mu(A)}_{\substack{<\frac{\varepsilon}{2} \\ \text{for } \mu(A)<\frac{\varepsilon}{2R}}} + \underbrace{\sup_n \int_{\{|f_n|>R\}} |f_n|\, d\mu}_{\overset{(iii)}{<} \frac{\varepsilon}{2} \text{ for } R \text{ big}} \overset{(iii)}{<} \varepsilon$$

$\Leftarrow$) $\mu(\{|f_n|>R\}) \overset{\text{Chebychev}}{\leq} \frac{\sup \|f_n\|_1}{R}$

$\overset{(i)}{\Rightarrow} \forall \varepsilon\ \exists R$ s.t. $\int_{|f_n|>R} |f_n| < \varepsilon \quad \forall n$

$\Rightarrow \lim_{R\to\infty} \sup_n \int_{|f_n|>R} |f_n| = 0$ $\square$

Generalization of LDCT:

<u>Lem. 49</u> (Vitali conv. thm.)

(X, μ) meas. sp., $\mu(X) < \infty$, $\{f_n\}$ equiint.,

$f_n \to f$ a.e., $|f(x)| < \infty$ a.e. (f, f_n meas.).

Then $f \in L^1(X)$ and

$$\| f_n - f \|_1 \xrightarrow[n \to \infty]{} 0.$$

<u>Rem. 50:</u>

$$\text{VCT} \Rightarrow \lim_{n \to \infty} \int f_n = \int f$$

VCT $\Rightarrow$ LDCT (since domination $\Rightarrow$ equiint)