Sobolevrom lec. 16

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OBS:
Send mail om
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19. Radon meas., $\mathcal{M} = (C_0)'$

Ref.: Folland: Real-anal. chp. 7; Holden

X loc. comp. Hausdorff (LCH) sp. (very general, see Holden)

Def. 58:

a) Radon meas. μ: A pos. Borel meas. on X satisfying

(i) $\mu(K) < \infty \quad \forall K \subset X$ comp. (loc. finite)

(ii) $\mu(B) = \inf\{\mu(U) : U \supset B,\ U \text{ open}\} \ \forall B \subset X$ Borel
(outer reg.)

(iii) $\mu(U) = \sup\{\mu(K) : K \subset U \text{ comp.}\} \ \forall U \subset X$ open
(inner reg.)

b) Signed Radon meas. μ: $\mu = \mu^+ - \mu^-$ ("Jordan decomp.")

and $\mu^{\pm}$ Radon meas.

c) $\mathcal{M}(X) := \{\mu : \mu \text{ signed Radon m.},\ \underbrace{|\mu|(X)}_{\text{tot. var.}} := (\mu^+ + \mu^-)(X) < \infty\}$

sp. of finite signed Radon meas.

d) $\mathcal{M}^+(X) := \{\mu : \mu \in \mathcal{M},\ \mu \geq 0\}$

Rem. 59:

a) (signed) Radon meas.: dx (Lebesgue); $f(x)\,dx$, $f \in L^1_{loc}$, δ-meas.

Not: δ'

b) $|f\,dx|(B) = \int_B |f|\,dx \quad \forall\, B \subset X$ meas., $f \in L^1$

$\left|\int f\,d\mu\right| \le \int_B |f|\,d|\mu|$, $\quad f \in L^1(B, |\mu|)$

$\mu \geq 0 \implies |\mu| = \mu$

μ (signed) Radon $\iff$ $|\mu|$ Radon

c) $(\mathcal{M}, \underbrace{|\cdot|(X)}_{\text{norm}})$ Banach sp.

[Normed sp.: Folland Prop. 7.16, Complete since $= \underbrace{(C_0(X))'}_{\text{compl.}}$]

Lin. func'als def. by Radon meas.:

$$\Phi_\mu(f) = \int f\,d\mu, \qquad \mu \text{ Radon}$$

Obs: $\Phi_\mu(f)$ well-def., finite for $f \in C_c(X)$

Φ_μ lin., $\|\Phi_\mu\| = \sup_{\substack{f \in C_c \\ \|f\|_\infty \le 1}} \left|\int_X f\,d\mu\right| < \infty \quad \overset{\text{[Fatou m. } f = (\chi_{\operatorname{supp}\mu^+ \cap B_R} - \chi_{\operatorname{supp}\mu^- \cap B_R})^\varepsilon \text{, outer + reg } \mu\text{, } R \to \infty,\ \varepsilon \to 0]}{\iff} \quad |\mu|(X) < \infty$

$\left|\int_X f\,d\mu\right| \le \|f\|_\infty |\mu|(X)$

Hence $\Phi_\mu \in (C_c)'$ iff $|\mu|(X) < \infty$.

Def. 60: $C_0(X) = \overline{C_c(X)}^{\|\cdot\|_\infty} = \{f \in C(X) : \exists f_n \in C_c,\ \|f - f_n\|_\infty \to 0\}$

Rem. 61:

a) $C_0(X) = \{ f \in C(X) : \forall \varepsilon > 0 \ \exists K_\varepsilon \subset X$ comp. st. $y \in X \setminus K_\varepsilon \Rightarrow |f(y)| < \varepsilon \}$

$C_0(X) \subset C_b(X)$

b) $(C_0, \|\cdot\|_\infty)$ Banach sp., separable when $X \subset \mathbb{R}^d$.

c) X comp. $\Rightarrow C(X) = C_c(X) = C_0(X)$

$\mu \in \mathcal{M}$ $(|\mu|(X) < \infty) :\Rightarrow \Phi_\mu$ well-def. on $C_0(X)$:

$$\Phi_\mu(f) := \lim_{n \to \infty} \Phi_\mu(f_n) \quad \text{where } \|\underset{\substack{\uparrow \\ C_c}}{f_n} - f\|_\infty \to 0$$

$[\,|\Phi_\mu(f_n) - \Phi_\mu(f_k)| \leq \|\Phi_\mu\| \cdot \|f_n - f_k\|_\infty$ Cauchy in $\mathbb{R}$ ...$]$

Hence $\Phi_\mu \in (C_0)' \ \forall \mu \in \mathcal{M}$, and

$$\mathcal{M} \subset (C_0)' \quad [\,\Phi(\mathcal{M}) \subset (C_0)'\,]$$

The opposite is also true:

Thm. 62: Riesz (-Markov) repr. thm.

X LCH, $\Phi : \mathcal{M}(X) \to (C_0(X))'$, $\mu \mapsto \Phi_\mu$.

Then Φ isometric isomorphism from $\mathcal{M}$ to $(C_0(X))'$.

Pf. Omitted

I.e. Φ invertible, lin., and

$$\|\mu\|_{\mathcal{M}} := |\mu|(X) = \|\Phi_\mu\| = \sup_{f \in C_0, \|f\|_\infty \leq 1} \left| \int f \, d\mu \right|$$

Obs: $|\int f \, d\mu| \le \|\mu\|_{\mathcal{M}} \cdot \|f\|_\infty \quad \forall f \in C_0$

We identify $\mathcal{M}$ and $C_0(X)'$:

$$\mathcal{M} = (C_0)' \qquad (\text{really: } \Phi(\mathcal{M}) = (C_0)')$$

W. * conv. in $\mathcal{M}$:

$$\mu_n \xrightarrow{*} \mu \text{ in } \mathcal{M} \overset{\text{def.}}{\Longleftrightarrow} \int_X f \, d\mu_n \to \int_X f \, d\mu \quad \forall f \in C_0$$

"(w. conv. in proba)"

<u>Ex. 63:</u>

$$f_n = n \chi_{(0,\frac{1}{n})} \in L^1(0,1) \quad [\|f_n\|_1 = 1]$$

$$\mu_n = f_n \, dx \in \mathcal{M}(0,1)$$

$$\int g \, d\mu_n = \int g \, f_n \, dx \longrightarrow g(0) \cdot 1 \quad \forall g \in C([0,1]) = C_0([0,1])$$

$$\Rightarrow \mu_n \xrightarrow{*} \delta_0 \text{ in } \mathcal{M}$$

But: $\|\mu_n - \delta_0\|_{\mathcal{M}} = (\mu_n - \delta_0)^+(X) + (\mu_n - \delta_0)^-(X) = \underbrace{\mu_n(X)}_{=\|f_n\|_1 = 1} + \delta_0(X) = 2$

so μ_n does not conv. str. in $\mathcal{M}$

Pause? 15:08

<u>Rem. 64:</u> $\|\mu_n - \mu\|_{\mathcal{M}} \to 0 \; \overset{\Longrightarrow}{\not\Leftarrow} \; \mu_n \xrightarrow{*} \mu$ in $\mathcal{M}$

[$\Rightarrow$) $|\int g \, d(\mu - \mu_n)| \le \|g\|_\infty \|\mu - \mu_n\|_{\mathcal{M}}$, $\not\Leftarrow$) Ex. 63]

20. <u>W. * seq'al comp. in $\mathcal{M}$</u>

s. 155 Simmonet: Meas and Proba

$X \subset \mathbb{R}^d$ Borel

<u>Thm. 65:</u> (w.$*$ seq'al comp. in $\mathcal{M}$)

$\mu_n \in \mathcal{M}(X)$, $\sup_n \|\mu_n\|_{\mathcal{M}} < \infty$

$\Rightarrow \exists\, \mu_{n_k}, \mu \in \mathcal{M}$ s.t. $\mu_{n_k} \xrightarrow{*} \mu$ in $\mathcal{M}$.

<u>Cor. 66:</u> ($\mathcal{M}$-comp. in L^1)

$\{f_n\} \subset L^1$, $\sup_n \|f_n\|_1 < \infty$

$\Rightarrow \exists\, f_{n_k}, \mu \in \mathcal{M}$ s.t.

$$\int g f_{n_k}\, dx \to \int f\, d\mu \quad \forall g \in C_0(X)$$

[$L^1(X, dx) \subset \mathcal{M}$: $\underset{L^1}{f} \mapsto \underset{\mathcal{M}}{f\,dx}$

$\sup_n \|f_n\, dx\|_{\mathcal{M}} = \sup_n \|f_n\|_1 < \infty$, use Thm. 65]

<u>Thm. 67:</u> (w.$*$ seq'al comp. in $(C_c)'$)

μ_n signed Radon meas. on X s.t.

$\sup_n |\mu_n|(K) < \infty \quad \forall\, K \subset X$ comp.

$\Rightarrow \exists\, \mu_{n_k}, \mu$ (signed Radon) s.t.

$$\int g\, d\mu_{n_k} \to \int g\, d\mu \quad \forall\, g \in C_c(X).$$

<u>Thm. 65 $\Leftrightarrow$ Thm. 67:</u>

$\Leftarrow$) Easy - HW

[Take $C_c \ni f_j \xrightarrow{L^\infty} f \in C_0$ + use $\sup_n |\mu_n|(X) < \infty$ ...]

6.)

(see Holder) $\Rightarrow$) Def. $\mu_n^m(B) = \mu_n(B \cap B(0,m))$

Thm. 65 + diag.-arg.

$\Downarrow$

$\exists\, \mu_{n_k}^{m_k}, \mu$ s.t.

$$\int g \, d\mu_{n_k}^{m_k} \longrightarrow \int g \, d\mu \quad \forall\, g \in \underset{\displaystyle C_0(\overline{B(0,m)})}{\overset{\shortparallel}{C(\overline{B(0,m)})}} \quad \forall m$$

Chk.: μ signed Radon ($\mu|_{B(0,m)} \in \mathcal{M}$ by Riesz) $\square$

Pf. of Thm. 75:

1) Subsequence:

$C_0(X)$ separable $\Rightarrow \exists \{f_j\}_j \subset C_0(X)$ dense, countable

By assumption:

$$\left|\int f_j \, d\mu_n\right| \leq \underbrace{\sup \|\mu_n\|_{\mathcal{M}}}_{=:M<\infty} \cdot \|f_j\|_\infty,$$

hence $\underbrace{}_{\text{by comp. in } \mathbb{R}}$ $\exists\, n_k, \ \alpha_j \in \mathbb{R}$ s.t.

$$\int f_j \, d\mu_{n_k} \xrightarrow[k\to\infty]{} \alpha_j$$

Diag. arg. gives further subseq'nce n_l and α_j, $j \in \mathbb{N}$, s.t.

$$\int f_j \, d\mu_{n_l} \longrightarrow \alpha_j \quad \forall j$$

2) Candidate limit:

Def. $\Lambda(f) = \lim\limits_{\|f_j - f\|_\infty \to 0} \Lambda(f_j) \quad \forall\, f \in C_0 = \overline{\{f_j\}_j}^{\|\cdot\|_\infty}$

$(\Lambda(f_j) = \alpha_j)$ 7.)

Chk.: $\Lambda \in (C_0)'$ $[\|\Lambda\| \leq M]$

$\overset{\text{Riesz}}{\Longrightarrow}$ $\exists \mu \in \mathcal{M}$ s.t. $\Lambda(f) = \int f \, d\mu \;\forall f \in C_0$

$[\mu = \Phi^{-1}\Lambda]$

3) Convergence:

Let $\varepsilon > 0$, $f \in C_0$ and $\{f_j\}_j \ni \tilde{f}_j \xrightarrow{L^\infty} f$.

Take J s.t. $\|f - \tilde{f}_j\|_\infty < \frac{\varepsilon}{M}$ for $j \geq J$

N s.t. $|\int \tilde{f}_J \, d\mu_n - \int \tilde{f}_J \, d\mu| < \varepsilon$ for $n \geq N$

Hence

$$|\int f \, d\mu_n - \int f \, d\mu|$$

$$\leq \underbrace{|\int (f - \tilde{f}_J) \, d\mu_n|}_{< \frac{\varepsilon}{M} \cdot M} + \underbrace{|\int \tilde{f}_J \, d\mu_n - \int \tilde{f}_J \, d\mu|}_{< \varepsilon} + \underbrace{|\int (\tilde{f}_J - f) \, d\mu|}_{< \frac{\varepsilon}{M} \cdot M}$$

$$< 3\varepsilon$$ $\square$

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