Sobolevrom lec 14 10.3.2019

Last time:

$\Omega \subset \mathbb{R}^d$ open, $m \in \mathbb{N}$, $p \in [1,\infty]$

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) : \underbrace{\partial^\alpha f}_{\text{w. deriv.}} \in L^p(\Omega) \ \forall |\alpha| \le m \}$$

$$\|f\|_{m,p} = \Big(\sum_{|\alpha| \le m} \|\partial^\alpha f\|_p^p \Big)^{\frac{1}{p}}$$

2. Smooth approx. (cont.)

A. Interior approx. (Thm. 5): $C^m(\Omega) \ni f_n \xrightarrow{W^{m,p}_{loc}} f \in W^{m,p}(\Omega)$

B. Global approx. (Thm. 6): $C^m(\Omega) \ni f_n \underset{(\Omega \text{ bnd})}{\xrightarrow{W^{m,p}}} f \in W^{m,p}(\Omega)$

C. Global up to the bnd'ry approx. by smooth func'ns

Approx. by $f_n \in C^m(\bar{\Omega})$

Need cond'ns on $\partial\Omega$ ("nice")

Thm. 8: $1 \le p < \infty$, $\Omega \subset \mathbb{R}^d$ open, bnd; $\partial\Omega$ is C^1; $f \in W^{m,p}(\Omega)$.

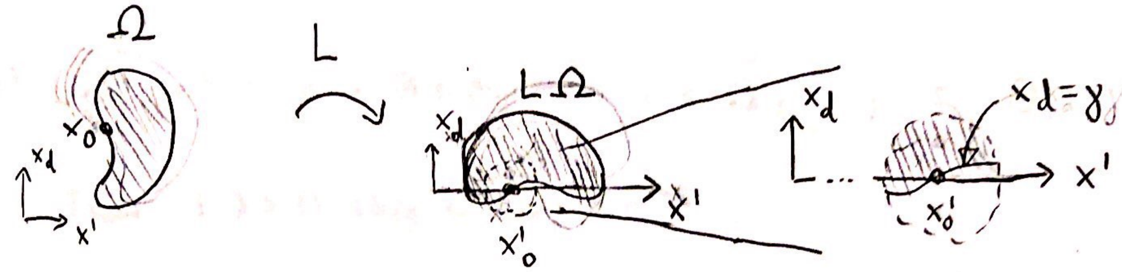
Then $\exists f_n \in C^\infty(\bar{\Omega})$ s.t. $\|f - f_n\|_{m,p} \underset{n \to \infty}{\longrightarrow} 0$.

Def. 9: $\Omega \subset \mathbb{R}^d$ is C^k ($\partial\Omega$ is C^k) if

$\forall x_0 \in \partial\Omega \quad \exists r > 0,\ \gamma \in C^k(\mathbb{R}^{d-1}),\ L$ transl. + rot. $(\det DL = 1)$
+ relabling (permutation)

s.t.

$$L\Omega \cap B((x_0',0),r) = \{ x = \underbrace{(x_1,\dots,x_d)}_{=(x',x_d)} \mid x_d > \gamma(\underbrace{x_1,\dots,x_{d-1}}_{x'}) \} \cap B((x_0',0),r)$$



Obs. 10:

a) In 2D: $L = \tau_{(x_0',0)} \circ R \circ \tau_{-x_0}$, $\quad L^{-1} = \tau_{x_0} \circ R^{-1} \circ \tau_{-(x_0',0)}$

$\uparrow$ rotation about 0 : $RR^T = I$

$$Lx = R(x - x_0) + (x_0', 0)$$

$$LB(x_0, r) = B((x_0',0), r)$$

"in general, relabel coord's if necessary"

b) L, L^{-1} vol. preserving, affine

$$D(Lx) = R,\quad D(L^{-1}y) = R^{-1}$$

$$\Rightarrow \det(DLx) = \det R = 1 = \dots = \det(DL^{-1}y)$$

c) $v(x) = u(L^{-1}x) \overset{\text{chk}}{\underset{\text{b)}}{\Rightarrow}} \|v\|_{L\Omega,m,p} = c_{R,m,p} \|u\|_{\Omega,m,p}$

$+ \nabla v = R^{-1} \cdot \nabla u \dots$

Pf. of Thm. 8: (Evans PDE)

1) Fix $x_0 \in \partial\Omega$

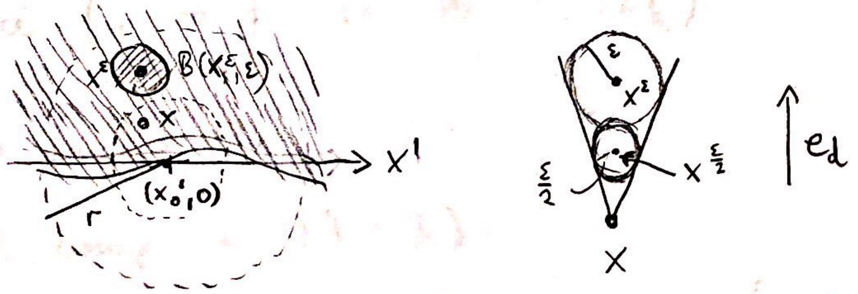
$\partial\Omega$ C^1 $\Rightarrow$ $\exists r > 0$, $\gamma \in C^1(\mathbb{R}^{d-1})$, L s.t.

$$\tilde{\Omega}_r := L\Omega \cap B((x_0', 0), r) = \{x \in B(\cdots, r) : x_d > \gamma(x')\}$$

2) Let $x^\varepsilon := x + \lambda\varepsilon e_d$, $x \in \tilde{\Omega}_{\frac{r}{2}}$, $\varepsilon > 0$.

Then $\exists \lambda > 0$ (big enough) s.t.

(*) $B(x^\varepsilon, \varepsilon) \subset \tilde{\Omega}_r$ $\forall$ $x \in \tilde{\Omega}_{\frac{r}{2}}$, $\varepsilon > 0$ small



[Ok since "interior cone cond'n holds in C^1 (or Lip.) domains:

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3) Def.: $g(x) := f(L^{-1}x)$, $g^\varepsilon(x) := g(x^\varepsilon)$, and

$$g_\varepsilon(x) := g^\varepsilon * \rho_\varepsilon(x) \quad \text{for } x \in \tilde{\Omega}_{\frac{r}{2}}, \quad (\varepsilon < \tfrac{r}{2})$$

where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, $0 \le \rho \in C_c^\infty$, $\int\rho = 1$, $\operatorname{supp}\rho \subset B(0,1)$

Def.: $f_\varepsilon(x) := g_\varepsilon(Lx)$, $x \in \Omega_{\frac{r}{2}} := \Omega \cap B(x_0, \frac{r}{2})$

Obs: Need g on $\tilde{\Omega}_r$ to def. g_ε on $\tilde{\Omega}_{\frac{r}{2}}$

Obs. 10 $''f_i$

$\Rightarrow$ $f_\varepsilon|_{\Omega_{\frac{r}{2}}}$ dep. only on $f|_{\Omega_r}$

$L\Omega_r = \tilde{\Omega}_r$

and is well-def.

15:14 Chk.: $f_\varepsilon \in C^\infty(\bar{\Omega}_{\frac{r}{2}})$ (easy)

$$\|\partial^\alpha f_\varepsilon - \partial^\alpha f\|_{\Omega_{\frac{r}{2}},p} \overset{\text{Obs. 10}}{=} c\|\partial^\alpha g_\varepsilon - \partial^\alpha g\|_{\tilde{\Omega}_{\frac{r}{2}},p}$$

$$\leq c\|\underbrace{\underset{\substack{\| \\ \rho_\varepsilon * \partial^\alpha g^\varepsilon}}{\partial^\alpha g_\varepsilon} - \partial^\alpha g^\varepsilon\|_p}_{\substack{\longrightarrow 0 \\ \text{by prev. res.}}} + \underbrace{c\|\partial^\alpha g^\varepsilon - \partial^\alpha g\|_p}_{\substack{\longrightarrow 0 \\ \text{by cont. of } L^p\text{-transl.} \\ \text{(ext. from } \Omega' \subset \Omega \text{ to } \mathbb{R}^d)}}$$

Hence $f_\varepsilon \to f$ in $W^{m,p}(\Omega_{\frac{r}{2}})$.

4.) Take any $\delta > 0$.

$\partial\Omega$ comp. + 1.) - 3.):

$\exists\, x_{0,1}, \dots, x_{0,N} \in \partial\Omega$, $r_i > 0$, func'ns f_i s.t.

$$\partial\Omega \subset \bigcup_{i=1}^N B(x_{0,i}, \tfrac{r_i}{2})$$

and

$f_i \in C^\infty(\bar{\Omega}_i)$, $\Omega_i = \Omega \cap B(x_{0,i}, \frac{r_i}{2})$,

$\|f_i - f\|_{W^{m,p}(\Omega_i)} < \delta$.

Take open Ω_0 s.t. $\bar{\Omega}_0 \subset \Omega$ and $f_0 \in C^\infty(\bar{\Omega}_0)$ s.t.

$\Omega \subset \bigcup_{i=0}^N \Omega_i$ and (by Thm. 5) $\|f_0 - f\|_{W^{m,p}(\Omega_0)} < \delta$.

5.) Take part. of 1 $\{\varphi_j\}_j$ subord. to $\Omega_0, \dots, \Omega_N$

Def.
$$\tilde{f} = \sum_{i=0}^{N} f_i \varphi_i \qquad (\operatorname{supp} \varphi_i \subset \Omega_i)$$

Obs: $\tilde{f} \in C^\infty(\overline{\Omega})$ and for $|\alpha| \leq m$,

$$\| \partial^\alpha \tilde{f} - \partial^\alpha f \|_{\Omega,p} \underset{f = \sum f \varphi_i}{\leq} \sum_{i=0}^{N} \| \partial^\alpha (f_i \varphi_i) - \partial^\alpha (f \varphi_i) \|_{\Omega_i,p}$$

$$\underset{\text{HW}}{\overset{\text{prod. rule}}{\leq}} C \sum_{i=0}^{N} \| \varphi_i \|_{\Omega_i, |\alpha|, \infty} \cdot \| f_i - f \|_{\Omega_i, |\alpha|, p} < \tilde{C} (N+1) \delta$$

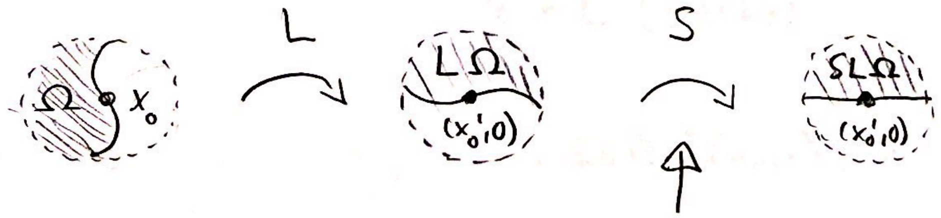
Hence $\| \tilde{f} - f \|_{\Omega, m, p} \leq \sum_{|\alpha| \leq m} \| \partial^\alpha \tilde{f} - \partial^\alpha f \|_{\Omega_i, p} < C_{m,N} \cdot \delta$ $\square$

3. Intermezzo: Straightening the bnd'ry

Useful tool to study funcns up to the bnd'ry.

"Reduce to (loc.) flat bnd'ry"

$\partial\Omega$ C^m; $x_0 \in \partial\Omega$; $r, \gamma \in C^m$, L given by Def. 9:



New: straightening

$$L\Omega \cap B((x_0', 0), r) = \{ x \in B((x_0', 0), r) : x_d > \gamma(x') \}$$

Def. 11: Straightening map

(i) $S : L\Omega \cap B(x_0', r) \to \mathbb{R}^d_+ = \{(x_1, \dots, x_d) : x_d > 0\}$

$$x \mapsto y = S(x) = (x', x_d - \gamma(x'))$$

(ii) $\Phi = S \circ L : \underbrace{\Omega \cap B(x_0, r)}_{\tilde{\Omega}} \to \mathbb{R}^d_+$

Obs. 12:

$x = S^{-1}(y) = (y', y_d + \gamma(y'))$

Chk.: $\det DS = 1 = \det DS^{-1}$ (Jacobian)

$S, S^{-1}, \Phi, \Phi^{-1} = L^{-1} \circ S^{-1}$ inv., vol. preserving

Thm. 13:

$\Psi : \bar{\Omega} \xrightarrow{C^m} \bar{G}$, $\Psi^{-1} : \bar{G} \xrightarrow{C^m} \bar{\Omega}$, $Au(x) = u(\Psi^{-1}(x))$

$\Rightarrow \|Au\|_{G,m,p} \leq c_1 \|u\|_{\Omega,m,p}$, $\|u\|_{\Omega,m,p} \leq c_2 \|Au\|_{G,m,p}$

[Adams-Fournier Thm. 3.41]

Obs: $\partial\Omega$ $C^m \Rightarrow \Phi \in C^m(\bar{\tilde{\Omega}})$ (Def. 11)

$\Phi^{-1} \in C^m(\overline{\Phi(\tilde{\Omega})})$

Cor. 14: $\partial\Omega$ C^m, $\tilde{\Omega} := \Omega \cap B(x_0, r)$, $u \in W^{m,p}(\tilde{\Omega})$,

$v(y) := u(\Phi^{-1}(y))$ for $y \in \Phi(\tilde{\Omega})$.

Then $v \in W^{m,p}(\Phi(\tilde{\Omega}))$ and

(i) $\|v\|_{\Phi(\tilde{\Omega}),p} = \|u\|_{\tilde{\Omega},p}$

(ii) $\frac{1}{c}\|u\|_{\tilde{\Omega},m,p} \leq \|v\|_{\Phi(\tilde{\Omega}),m,p} \leq c\|u\|_{\tilde{\Omega},m,p}$

Pf.: Thm. 13 w. $\Psi = \Phi$, $G = \Phi(\tilde{\Omega})$

HW: Do $|\alpha| = 1$ case. $\square$

Hint: $\partial_i(u(\Phi^{-1}(x))) = \sum u_{x_j} \cdot \partial_i \Phi^{-1}_j$, $\det D\Phi = 1$ $\square$