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Solutions to exercise set 1

1 By strict convexity of the exponential map it follows that

$$ab = \exp\left(\tfrac{1}{p}\log(a^p) + \tfrac{1}{q}\log(b^q)\right) < \tfrac{1}{p}\exp\log(a^p) + \tfrac{1}{q}\exp\log(b^q) = \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $\log(a^p) = \log(b^q)$, that is, if and only if $a^p = b^q$. (This is valid for all $p, q \in (1,\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.)

2 Doing nothing (multiplying by 1) in a clever way in Young's inequality:

$$ab = \left(\epsilon^{1/p} a\right)\left(\epsilon^{1/q - 1} b\right) \leq \epsilon\frac{a^p}{p} + \frac{b^q}{q\epsilon^{q-1}}.$$

3 By rescaling $x \mapsto x/\|x\|_p$ and $y \mapsto y/\|y\|_q$, it suffices to establish that

$$\left|\sum_n x_n y_n\right| \leq 1 \qquad \text{for all} \qquad x \in \ell^p \text{ and } y \in \ell^q \quad \text{with} \quad \|x\|_p = 1 = \|y\|_q.$$

Now the claim follows directly from Young's inequality:

$$\left|\sum_n x_n y_n\right| \leq \sum_n |x_n y_n| \leq \sum_n \left(\tfrac{1}{p}|x_n|^p + \tfrac{1}{q}|y_n|^q\right) = \tfrac{1}{p}\|x\|_p^p + \tfrac{1}{q}\|y\|_q^q = \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

4 The given statement should have been in the other direction. ℓ^p-spaces are *increasing* with $p \in [1,\infty]$, that is, $\ell^p \subset \ell^q$ for $1 \leq p < q \leq \infty$, because elements in ℓ^p have to decay faster at infinity than those in ℓ^q (in order for the sums to converge). In fact,

$$\ell^p \hookrightarrow \ell^q \qquad \text{with} \qquad \|x\|_q \leq \|x\|_p \quad \text{for all} \quad 1 \leq p < q \leq \infty.$$

Proof: Case $q = \infty$ is clear, so assume $q < \infty$. By rescaling, it is enough to establish $\|x\|_q \leq 1$ for $x \in \ell^p$ with $\|x\|_p = 1$. For such x we must especially have $|x_n| \leq 1$ for all n. Therefore $|x_n|^q \leq |x_n|^p$ since $q > p$. Hence,

$$\|x\|_q^q = \sum_n |x_n|^q \leq \sum_n |x_n|^p = \|x\|_p^p = 1,$$

and the claim follows.

Note: $L^p(\Omega)$-spaces, however, are *decreasing* with $p \in [1, \infty]$, provided Ω has finite measure $|\Omega| < \infty$:

$$L^q(\Omega) \hookrightarrow L^p(\Omega) \qquad \text{with} \qquad \|f\|_p \le |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|f\|_q \quad \text{for all} \quad 1 \le p < q \le \infty.$$

(A straightforward application of Hölder's inequality.) This is in general not true for sets of infinite measure, and in particular, does not hold for $\mathbb{R}^n$. But, if $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$ for some $1 \le p_1 < p_2 \le \infty$, with no restrictions on Ω, then $f \in L^p(\Omega)$ for all $p \in [p_1, p_2]$. This is an example of real interpolation and also follows from Hölder's inequality.

$\boxed{5}$ Canonically, the map $x \mapsto \varphi_x$ where φ_x acts as $\langle \varphi_x, y\rangle = \sum_n x_n y_n$ defines an isometric isomorphism $\ell^p \cong (\ell^q)'$ for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, φ_x is linear, and bounded and well-defined by Hölder: $|\langle \varphi_x, y\rangle| \le \|x\|_p \|y\|_q$. Thus $\|\varphi_x\|_{(\ell^q)'} \le \|x\|_p$, but in fact we have equality: define y by

$$y_n = \begin{cases} x_n^{-1}|x_n|^p & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0. \end{cases}$$

Then $y \in \ell^q$ with $\|y\|_q = \|x\|_p^{p/q}$, and $\langle \varphi_x, y\rangle = \|x\|_p^p$. Since $p - \frac{p}{q} = 1$ by assumption, this yields $\|\varphi_x\|_{(\ell^q)'} \ge |\langle \varphi_x, y\rangle| / \|y\|_q = \|x\|_p$. As such, the canonical map is an isometry, and in particular, injective.

For surjectivity, we need some notation. Let e_k denote the sequence with a 1 in the kth coordinate and 0 elsewhere. Then every sequence y has the representation $y = \sum_k y_k e_k$. Now let $y' \in (\ell^q)'$ and construct x as $x_n = \langle y', e_n\rangle$. Is $x \in \ell^p$? Yes, for all $N \in \mathbb{N}$, using that $p = \frac{p}{q} + 1$, we have

$$\begin{aligned}
\sum_{n=0}^{N} |x_n|^p &= \sum_{n=0}^{N} \operatorname{sgn}(x_n)|x_n|^{p/q} \langle y', e_n\rangle \\
&= \left\langle y', \sum_{n=0}^{N} \operatorname{sgn}(x_n)|x_n|^{p/q} e_n \right\rangle \\
&\le \|y'\|_{(\ell^q)'} \left\| \sum_{n=0}^{N} \operatorname{sgn}(x_n)|x_n|^{p/q} e_n \right\|_q \\
&= \|y'\|_{(\ell^q)'} \left(\sum_{n=0}^{N} |x_n|^p \right)^{1/q}
\end{aligned}$$

Rearranging gives $\left(\sum_{n=0}^{N} |x_n|^p\right)^{1/p} \le \|y'\|_{(\ell^q)'}$, and we can pass to the limit $N \to \infty$. Moreover,

$$\langle \varphi_x, e_k\rangle = \sum_n x_n (e_k)_n = x_k = \langle y', e_k\rangle,$$

which, by linearity, implies that $\langle \varphi_x, y\rangle = \langle y', y\rangle$ for all $y \in \ell^q$. Hence, $\varphi_x = y'$.