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MA8105
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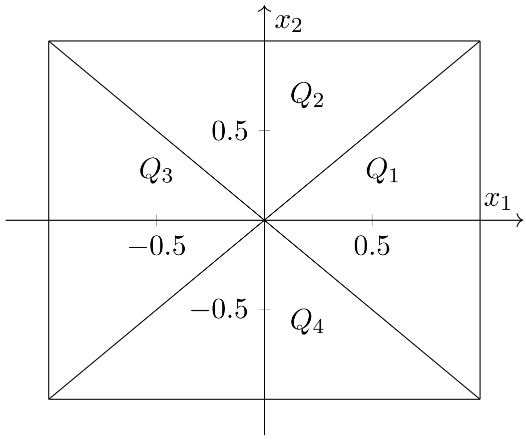
Solutions to exercise set 10

1 **(Evans, problem 3 p. 306):** Let $Q = (-1,1)^2 \subset \mathbb{R}^2$ and

$$f(x) = \begin{cases} 1 - x_1, & x \in Q_1 := \{x \mid x_1 > 0, |x_2| < x_1\} \\ 1 + x_1, & x \in Q_3 := \{x \mid x_1 < 0, |x_2| < -x_1\} \\ 1 - x_2, & x \in Q_2 := \{x \mid x_2 > 0, |x_1| < x_2\} \\ 1 + x_2, & x \in Q_4 := \{x \mid x_2 < 0, |x_1| < -x_2\} \end{cases}$$



We are to show that $f \in W^{1,p}(Q)$ for $p \in [1,\infty]$. First note that $f(x) \le 1$ for $x \in Q$ so that $\|f\|_{Q,0,\infty} \le 1 < \infty$ and $\|f\|_{Q,0,p} \le \lambda(Q)^{1/p} = 4^{1/p} < \infty$ for $p \in [1,\infty)$. Hence $f \in L^p(Q)$ for $p \in [1,\infty]$, and it remains to verify the same for its weak derivatives. Take any $\varphi \in C_c^\infty(Q)$, then

$$\begin{aligned} \int_Q (\partial_{x_1} f(x))\varphi(x)\,dx &:= -\int_Q f(x)\partial_{x_1}\varphi(x)\,dx = -\sum_{j=1}^4 \int_{Q_j} f(x)\partial_{x_1}\varphi(x)\,dx \\ &= \sum_{j=1}^4 \int_{Q_j} (\partial_{x_1} f(x))\varphi(x)\,dx = \int_Q (1_{Q_3}(x) - 1_{Q_1}(x))\varphi(x)\,dx. \end{aligned}$$

Note the lack of boundary integrals over ∂Q_j in the above, as these cancel out due to the continuity of f along the boundaries of Q_j. Thus $1_{Q_3} - 1_{Q_1}$ is a weak derivative of f with respect to x_1, and similarly we find that $1_{Q_4} - 1_{Q_2}$ is a weak derivative with respect to x_2. As indicator functions of sets in Q belong to $L^p(Q)$ for $p \in [1,\infty]$ we have our result.

2 First, we assume that $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ are open and bounded sets, and $\Phi : \overline{\Omega}_1 \to \overline{\Omega}_2$ and $\Phi^{-1} : \overline{\Omega}_2 \to \overline{\Omega}_1$ are invertible C^m-transforms. By the regularity of Φ and Φ^{-1}, and the compactness of $\overline{\Omega}_1$ and $\overline{\Omega}_2$, we have

$$\frac{1}{\lambda} \leq |\det(D\Phi)| + |\det(D\Phi^{-1})| \leq \lambda$$
$$\|\Phi\|_{W^{1,\infty}(\overline{\Omega}_1)}, \|\Phi^{-1}\|_{W^{1,\infty}(\overline{\Omega}_2)} < \infty$$

where $\lambda > 1$. To obtain the estimate of the $W^{1,p}$-norm of g, we apply the chain rule, Minkowski's inequality, Hölder's generalized inequality, the standard change-of variables formula, and the upper bound $|\det(D\Phi)| \leq \lambda$:

$$\begin{aligned}
&\|g\|^p_{W^{1,p}(\Omega_2)} \\
&= \sum_{|\alpha|\leq 1} \|\partial^\alpha \left[f(\Phi^{-1}(\mathbf{x}))\right]\|^p_{L^p(\Omega_2)} \\
&= \|f(\Phi^{-1}(\mathbf{x}))\|^p_{L^p(\Omega_2)} + \sum_{i=1}^d \left\| \sum_{j=1}^d \frac{\partial f(\Phi^{-1}(\mathbf{x}))}{\partial x_i} \frac{\partial \Phi_j^{-1}}{\partial x_i} \right\|^p_{L^p(\Omega_2)} \\
&\leq \|f(\Phi^{-1}(\mathbf{x}))\|^p_{L^p(\Omega_2)} + \sum_{i=1}^d \left(\sum_{j=1}^d \left\| \frac{\partial f(\Phi^{-1}(\mathbf{x}))}{\partial x_i} \frac{\partial \Phi_j^{-1}}{\partial x_i} \right\|_{L^p(\Omega_2)} \right)^p \\
&\leq \|f(\Phi^{-1}(\mathbf{x}))\|^p_{L^p(\Omega_2)} + \sum_{i=1}^d \left\| \frac{\partial f(\Phi^{-1}(\mathbf{x}))}{\partial x_i} \right\|^p_{L^p(\Omega_2)} \left(\sum_{j=1}^d \left\| \frac{\partial \Phi_j^{-1}}{\partial x_i} \right\|_{L^\infty(\Omega_2)} \right)^p \\
&\leq \|f(\Phi^{-1}(\mathbf{x}))\|^p_{L^p(\Omega_2)} + d^p \|\Phi^{-1}\|^p_{W^{1,\infty}} \sum_{i=1}^d \left\| \frac{\partial f(\Phi^{-1}(\mathbf{x}))}{\partial x_i} \right\|^p_{L^p(\Omega_2)} \\
&\leq \underbrace{\max\{1, d^p \|\Phi^{-1}\|^p_{W^{1,\infty}}\}}_{C_1^p} \sum_{|\alpha|\leq 1} \|\partial^\alpha f(\Phi^{-1}(\mathbf{x}))\|^p_{L^p(\Omega_2)} \\
&= C_1^p \sum_{|\alpha|\leq 1} \int_{\Omega_2} \left|\partial^\alpha f(\Phi^{-1}(\mathbf{x}))\right|^p \, d\mathbf{x} \\
&= C_1^p \sum_{|\alpha|\leq 1} \int_{\Omega_1} |\partial^\alpha f(\mathbf{y})|^p \, |\det(D\Phi)| \, d\mathbf{y} \\
&\leq C_1^p \sum_{|\alpha|\leq 1} \int_{\Omega_1} |\partial^\alpha f(\mathbf{y})|^p \, \lambda \, d\mathbf{y} \\
&\leq C_1^p \lambda \|f\|^p_{W^{1,p}(\Omega_1)}
\end{aligned}$$

Thus, by taking the p-root on both sides of the inequality, we get

$$\|g\|_{W^{1,p}(\Omega_2)} \leq C_1 \lambda^{1/p} \|f\|_{W^{1,p}(\Omega_1)} \tag{2}$$

We do the same in the opposite direction, with the upper bound $|\det(D\Phi^{-1})| \leq \lambda$:

$$\begin{aligned}
&\|f\|^p_{W^{1,p}(\Omega_1)} \\
&= \sum_{|\alpha|\leq 1} \|\partial^\alpha \left[g(\Phi(\mathbf{y}))\right]\|^p_{L^p(\Omega_1)} \\
&= \|g(\Phi(\mathbf{y}))\|^p_{L^p(\Omega_1)} + \sum_{i=1}^d \left\| \sum_{j=1}^d \frac{\partial g(\Phi(\mathbf{y}))}{\partial y_i} \frac{\partial \Phi_j}{\partial y_i} \right\|^p_{L^p(\Omega_1)} \\
&\leq \|g(\Phi(\mathbf{y}))\|^p_{L^p(\Omega_1)} + \sum_{i=1}^d \left(\sum_{j=1}^d \left\| \frac{\partial g(\Phi(\mathbf{y}))}{\partial y_i} \frac{\partial \Phi_j}{\partial y_i} \right\|_{L^p(\Omega_1)} \right)^p \\
&\leq \|g(\Phi(\mathbf{y}))\|^p_{L^p(\Omega_1)} + \sum_{i=1}^d \left\| \frac{\partial g(\Phi(\mathbf{y}))}{\partial y_i} \right\|^p_{L^p(\Omega_1)} \left(\sum_{j=1}^d \left\| \frac{\partial \Phi_j}{\partial y_i} \right\|_{L^\infty(\Omega_1)} \right)^p \\
&\leq \|g(\Phi(\mathbf{y}))\|^p_{L^p(\Omega_1)} + d^p \|\Phi\|^p_{W^{1,\infty}} \sum_{i=1}^d \left\| \frac{\partial g(\Phi(\mathbf{y}))}{\partial y_i} \right\|^p_{L^p(\Omega_1)} \\
&\leq \underbrace{\max\{1, d^p\|\Phi\|^p_{W^{1,\infty}}\}}_{C_2^p} \sum_{|\alpha|\leq 1} \|\partial^\alpha g(\Phi(\mathbf{y}))\|^p_{L^p(\Omega_1)} \\
&= C_2^p \sum_{|\alpha|\leq 1} \int_{\Omega_1} |\partial^\alpha g(\Phi(\mathbf{y}))|^p \, d\mathbf{y} \\
&= C_2^p \sum_{|\alpha|\leq 1} \int_{\Omega_2} |\partial^\alpha g(\mathbf{x})|^p \, |\det(D\Phi^{-1})| \, d\mathbf{x} \\
&\leq C_2^p \sum_{|\alpha|\leq 1} \int_{\Omega_2} |\partial^\alpha g(\mathbf{x})|^p \, \lambda \, d\mathbf{x} \\
&\leq C_2^p \lambda \|g\|^p_{W^{1,p}(\Omega_2)}
\end{aligned}$$

Thus, by taking the p-root on both sides of the inequality, we get

$$\|f\|_{W^{1,p}(\Omega_1)} \leq C_2 \lambda^{1/p} \|g\|_{W^{1,p}(\Omega_2)} \tag{3}$$

The last step is to combine inequality (2) and (3). By setting $c = \lambda^{1/p}\max\{C_1, C_2\}$, we obtain the final estimate:

$$\frac{1}{c}\|f\|_{W^{k,p}(\Omega_2)} \leq \|g\|_{W^{1,p}(\Omega_2)} \leq c\|f\|_{W^{k,p}(\Omega_1)} \tag{4}$$

$\boxed{3}$ Let $u \in W^{1,p}(\widetilde{\Omega})$ such that $\widetilde{\Omega} = \Omega \cap B(\mathbf{x}_0, r)$ and $\partial\Omega \in C^1$. Since $|\det(D\Phi)| = 1$, we get $|\det(D\Phi^{-1})| = 1$. If $v(\mathbf{y}) = u(\Phi^{-1}(\mathbf{y}))$ for all $\mathbf{y} \in \Phi(\widetilde{\Omega})$, and $\mathbf{y} = \Phi(\mathbf{x})$, then

$$\begin{aligned}
\|v\|^p_{L^p(\Phi(\widetilde{\Omega}))} &= \int_{\Phi(\widetilde{\Omega})} |v(\mathbf{y})|^p \, d\mathbf{y} \\
&= \int_{\widetilde{\Omega}} |u(\Phi^{-1}(\mathbf{y}))|^p |\det(D\Phi)| \, d\mathbf{y} \\
&= \int_{\widetilde{\Omega}} |u(\mathbf{x})|^p \, d\mathbf{x} \\
&= \|u\|^p_{L^p(\widetilde{\Omega})}
\end{aligned}$$

By using the chain rule and assuming that $|\alpha| = 1$ and $\Omega \subset \mathbb{R}^d$, we get

$$\begin{aligned}
\|\partial^\alpha v\|^p_{L^p(\Phi(\widetilde{\Omega}))} &= \int_{\Phi(\widetilde{\Omega})} |\partial^\alpha v(\mathbf{y})|^p \, d\mathbf{y} \\
&= \int_{\widetilde{\Omega}} |\partial^\alpha [u(\Phi^{-1}(\mathbf{y}))]|^p |\det(D\Phi)| \, d\mathbf{y} \\
&= \int_{\widetilde{\Omega}} \left| \sum_{i=1}^d \partial^\alpha u(\Phi^{-1}(\mathbf{y})) \partial^\alpha \Phi_i^{-1}(\mathbf{y}) \right|^p \, d\mathbf{y} \\
&\leq \int_{\widetilde{\Omega}} |\partial^\alpha u(\Phi^{-1}(\mathbf{y}))|^p \left| \sum_{i=1}^d \partial^\alpha \Phi_i^{-1}(\mathbf{y}) \right|^p \, d\mathbf{y} \\
&\leq \|\partial^\alpha u^p\|_{L^1} \|[\partial^\alpha \Phi^{-1}]^p\|_{L^\infty} \\
&= \|u\|^p_{W^{1,p}} \|\Phi^{-1}\|^p_{W^{\infty,p}}
\end{aligned}$$

Similarly, in the opposite direction, we get

$$\begin{aligned}
\|\partial^\alpha u\|^p_{L^p(\widetilde{\Omega})} &= \int_{\widetilde{\Omega}} |\partial^\alpha u(\mathbf{x})|^p \, d\mathbf{x} \\
&= \int_{\Phi(\widetilde{\Omega})} |\partial^\alpha [v(\Phi(\mathbf{x}))]|^p \det(D\Phi^{-1})| \, d\mathbf{x} \\
&= \int_{\Phi(\widetilde{\Omega})} \left| \sum_{i=1}^d \partial^\alpha v(\Phi(\mathbf{x})) \partial^\alpha \Phi_i(\mathbf{x}) \right|^p \, d\mathbf{x} \\
&\leq \int_{\widetilde{\Omega}} |\partial^\alpha v(\Phi(\mathbf{x}))|^p \left| \sum_{i=1}^d \partial^\alpha \Phi_i(\mathbf{x}) \right|^p \, d\mathbf{x} \\
&\leq \|\partial^\alpha v^p\|_{L^1} \|[\partial^\alpha \Phi]^p\|_{L^\infty} \\
&= \|v\|^p_{W^{1,p}} \|\Phi\|^p_{W^{\infty,p}}
\end{aligned}$$

We have demonstrated that the $W^{k,p}$-norm of v is bounded by u, which belongs to $W^{k,p}$. Thus, $v \in W^{k,p}$, and we have shown that

$$\|v\|_{L^p(\Phi(\widetilde{\Omega}))} = \|u\|_{L^p(\widetilde{\Omega})} \tag{5a}$$

$$\|\partial^\alpha v\|_{L^p(\Phi(\widetilde{\Omega}))} \leq \|\Phi^{-1}\|_{W^{1,\infty}} \|u\|_{W^{1,p}} \tag{5b}$$

$$\|\partial^\alpha u\|_{L^p(\widetilde{\Omega})} \leq \|\Phi\|_{W^{1,\infty}} \|v\|_{W^{1,p}} \tag{5c}$$

4 **(Evans, problem 7 p. 306):** Given $1 \leq p < \infty$, a subset $\Omega \subset \mathbb{R}^d$ which is open and bounded, and a C^1 vector field γ along $\partial\Omega$ such that $\gamma \cdot \mathbf{n} \geq 1$ where $\mathbf{n}$ is the normal vector pointing outward from Ω. We are to apply the divergence theorem to $\int_{\partial\Omega} |f|^p (\gamma \cdot \mathbf{n})\, dS$ to derive the trace inequality

$$\int_{\partial\Omega} |f|^p\, dS \leq C \int_\Omega (|\nabla f|^p + |f|^p)\, dx.$$

Here we assume for simplicity $f \in C^1(\bar{\Omega})$ and if this were not the case we could have done an approximation argument. We go to work and obtain

$$\begin{aligned}
&\int_{\partial\Omega} |f|^p\, dS \overset{\text{by assumption}}{\leq} \int_{\partial\Omega} (|f|^p \gamma) \cdot \mathbf{n}\, dS \overset{\text{div. thm.}}{=} \int_\Omega \nabla \cdot (|f|^p \gamma)\, dx \\
&= \int_\Omega [(\nabla |f|^p) \cdot \gamma + |f|^p (\nabla \cdot \gamma)]\, dx = \int_\Omega \left[p|f|^{p-1} (\nabla f \cdot \gamma) + |f|^p (\nabla \cdot \gamma) \right] dx \\
&\leq \int_\Omega \left[p|f|^{p-1} |\nabla f| |\gamma| + |f|^p \sum_{j=1}^d \|\partial_{x_j} \gamma_j\|_{\Omega,0,\infty} \right] dx \\
&\leq \int_\Omega \left[p|f|^{p-1} |\nabla f| \sum_{j=1}^d \|\gamma_j\|_{\Omega,0,\infty} + |f|^p \sum_{j=1}^d \|\partial_{x_j} \gamma_j\|_{\Omega,0,\infty} \right] dx \\
&\overset{\text{Young}}{\leq} \int_\Omega \left[pd \max_{1 \leq j \leq d} \|\gamma_j\|_{\Omega,0,\infty} \left(\frac{p-1}{p} |f|^p + \frac{1}{p} |\nabla f|^p \right) + d \max_{1 \leq j \leq d} \|\partial_{x_j} \gamma_j\|_{\Omega,0,\infty} |f|^p \right] dx \\
&\leq d \max_{1 \leq j \leq d} \left\{ \|\gamma_j\|_{\Omega,0,\infty}, \|\partial_{x_j} \gamma_j\|_{\Omega,0,\infty} \right\} \int_\Omega (|\nabla f|^p + |f|^p)\, dx,
\end{aligned}$$

which is what we wanted.