Norwegian University of Science and Technology
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MA8105

Nonlinear PDEs and Sobolev spaces
Spring 2019

Exercise set 10

1 (Evans P3 p 306) Let $Q = (-1,1)^2$ (open square) and let f be the tent function (pyramid) supported on Q defined by

$$f(x) = \begin{cases} 1 - x_1 & \text{for } x_1 > 0,\ x_1 > |x_2|, \\ 1 + x_1 & \text{for } x_1 < 0,\ -x_1 > |x_2|, \\ 1 - x_2 & \text{for } x_2 > 0,\ x_2 > |x_1|, \\ 1 + x_2 & \text{for } x_2 < 0,\ -x_2 > |x_1|. \end{cases}$$

Show that $f \in W^{1,p}(Q)$ for all $p \in [1, \infty]$.

Hint: Find explicitly the weak derivative.

2 Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be bounded open and $\Phi : \overline{\Omega}_1 \to \overline{\Omega}_2$ and $\Phi^{-1} : \overline{\Omega}_2 \to \overline{\Omega}_1$ be invertible C^m transformations.

If $f \in W^{m,p}(\Omega_1)$, show that then $g(x) = f(\Phi^{-1}(x)),\ x \in \Omega_2$, belongs to $W^{m,p}(\Omega_2)$ and that there is $c > 1$ such that

$$\frac{1}{c}\|f\|_{\Omega_1,m,p} \leq \|g\|_{\Omega_2,m,p} \leq c\|f\|_{\Omega_1,m,p}.$$

Hint: Chain rule + change of variables in multiple integral formula (involving e.g. $\det(D\Phi)$) + only do proof for $m = 1$ where you may assume (by invertibility and continuity) that there is $\lambda > 1$ such that

$$\frac{1}{\lambda} \leq |\det D\Phi| + |\det(D\Phi^{-1})| \leq \lambda.$$

3 Prove Lemma 13 from the lectures when $m = 1$ (see scan under "Lectures" on the webpage).

Hint: Use exercise 2 and Obs 10 from the lectures.

4 (Evans P7 p 306) Assume $1 \leq p < \infty$, $\Omega \subset \mathbb{R}$ open, bounded, and there exists a C^1 vector field γ along $\partial\Omega$ such that $\gamma \cdot n \geq 1$ where n is the outward unit normal.

Apply the divergence theorem to $\int_{\partial\Omega} |f|^p \gamma \cdot n\, dS$ to derive a new proof of the trace inequality

$$\int_{\partial\Omega} |f|^p dS \leq C \int_\Omega (|Df|^p + |f|^p) dx.$$