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Solutions to exercise set 11

1 Let $\tilde{C}$ be the constant from the Poincare inequality. Then we get

$$\begin{aligned}|f|_{1,p}^p = \int |\nabla f|^p \leq \int |f|^p + \int |\nabla f|^p = ||f||_{1,p}^p \\ \leq \tilde{C}^p |f|_{1,p}^p + |f|_{1,p}^p = (1+\tilde{C}^p)|f|_{1,p}^p.\end{aligned}$$

Taking the p'th root we see that $C = \left(1+\tilde{C}^p\right)^{1/p}$.

2 **(Evans, problem 8 p. 308):** We are to show that there are in general no trace operator for L^p-functions ($1 \leq p < \infty$), i.e. a bounded linear operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tf = f\big|_{\partial\Omega}$ for $f \in L^p(\Omega) \cap C(\bar{\Omega})$. It is enough to find one example such that this cannot hold, so let us consider $\Omega = B(0,1) \subset \mathbb{R}^2$ and define

$$f_n(r,\theta) = ((1-n(1-r))^+)^{1/p},$$

where by g^+ we mean the positive part of a function g. It is clear that $0 \leq f_n \leq 1$ and $f_n \in C(\bar{\Omega})$. It is straightforward to show that $\|f_n\|_{L^p(\partial\Omega)}^p = 2\pi$, while further calculations yield

$$\|f_n\|_{L^p(\Omega)}^p = \int_0^{2\pi} \int_{1-1/n}^1 (1-n(1-r))r\,dr\,d\theta = \cdots = \frac{2\pi}{6n^2}(3n-1).$$

Together this gives

$$\frac{\|f_n\|_{L^p(\partial\Omega)}}{\|f_n\|_{L^p(\Omega)}} = \left(\frac{6n}{3-\frac{1}{n}}\right)^{1/p} \xrightarrow{n\to\infty} \infty,$$

i.e. there can be no constant C such that $\|f_n\|_{L^p(\partial\Omega)} \leq C\|f_n\|_{L^p(\Omega)}$ for all n, and so T cannot be bounded if it exists.

3 **(Evans, problem 4 p. 306):** We are given $f \in W^{1,p}((0,1))$ for $p \in [1,\infty)$, and are to show that

$$|f(x)-f(y)| \leq \left(\int_0^1 |f'(t)|^p\,dt\right)^{1/p} |x-y|^{1-1/p}. \tag{1}$$

As $f \in W^{1,p}((0,1))$ we know that its weak derivative f' is defined a.e. and belongs to $L^p((0,1))$. Let us then define $g(x) = \int_0^x f'(x)\,dx$ which is well defined as $L^p((0,1)) \subseteq L^1((0,1))$, and even absolutely continuous as it is defined as an integral from 0 to x of a integrable function. From the first fundamental theorem of calculus we have that $g'(x) = f'(x)$ a.e., and then for any $\varphi \in C_c^\infty((0,1))$ we have

$$\int_0^1 (f(x)-g(x))\varphi'(x)\,dx = -\int_0^1 (f'(x)-g'(x))\varphi(x)\,dx = 0,$$

showing that the weak derivatives of f and g are equal a.e. and then the functions themselves a.e. differ by at most a constant. Then for x, y where f is defined we have

$$\begin{aligned}
|f(x) - f(y)| = |g(x) - g(y)| = \left| \int_0^x f'(t)\, dt - \int_0^y f'(t)\, dt \right| &\le \int_{(0,x)\triangle(0,y)} |f'(t)|\, dt \\
&\overset{\text{Hölder}}{\le} \left(\int_{(0,x)\triangle(0,y)} |f'(t)|^p\, dt \right)^{1/p} |x-y|^{1-1/p} \\
&\le \left(\int_0^1 |f'(t)|^p\, dt \right)^{1/p} |x-y|^{1-1/p},
\end{aligned}$$

which is what we wanted.

$\boxed{4}$ **a)** By Greens first identity and zero boundary terms because of compact support

$$\begin{aligned}
\int_\Omega |\nabla f|^p = \int_\Omega \nabla f \cdot \nabla f |\nabla f|^{p-2} &= -\int_\Omega f\, \nabla \cdot \left(\nabla f |\nabla f|^{p-2} \right) \\
&= -\int_\Omega f \triangle f |\nabla f|^{p-2} - \int_\Omega f\, \nabla f \cdot \nabla \left(|\nabla f|^{p-2} \right)
\end{aligned}$$

For the last term we have

$$\frac{\partial}{\partial x_i} |\nabla f|^{p-2} = (p-2)|\nabla f|^{p-4} \sum_{j=1}^d f_{x_j} f_{x_i x_j} = (p-2)|\nabla f|^{p-4} (\nabla^2 f\, \nabla f)_i$$

where $\nabla^2 f$ is the Hessian matrix of f. Hence

$$\begin{aligned}
\int_\Omega |\nabla f|^p = -\int_\Omega f \triangle f |\nabla f|^{p-2} - (p-2) \int_\Omega f |\nabla f|^{p-4} (\nabla f)^T \nabla^2 f\, \nabla f. \\
\le C \int_\Omega f |\nabla f|^{p-2} |\nabla^2 f|.
\end{aligned}$$

At this point we want to use Holders inequality. We need to make the exponents for $|\nabla f|$ on both sides of the equation match.

$$\int_\Omega |\nabla f|^p \le C \left(\int_\Omega |\nabla f|^p \right)^{1-\frac{2}{p}} \left(\int_\Omega |f|^{\frac{p}{2}} |\nabla^2 f|^{\frac{p}{2}} \right)^{\frac{2}{p}}.$$

Notice that $p \ge 2$ is required. Dividing and applying Cauchy Schwartz inequality gives

$$\int_\Omega |\nabla f|^p \le \tilde{C} \left(\int_\Omega |f|^p \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla^2 f|^p \right)^{\frac{1}{2}}.$$

Taking the p'th root gives the desired estimate.

b) First we claim that if $f \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ then there are

$$\begin{aligned}
f_m \in C^2(\bar{\Omega}), f_m \to f \quad &\text{in } W^{2,2} \\
g_m \in C_c^\infty, g_m \to g \quad &\text{in } W^{1,2}.
\end{aligned}$$

The claims follows since $\partial\Omega$ is C^2. The first claim is a consequence of theorem 8 since $C^\infty(\bar{\Omega}) \subset C^2(\bar{\Omega})$. The second claim follows by theorem 21, the second trace theorem.

We use Greens identity on $\nabla f_m \cdot \nabla g_m$ in a way so that the boundary terms gives zero contribution ($g_m \in C_c^\infty(\Omega)$)

$$\int_\Omega \nabla f_m \cdot \nabla g_m = -\int_\Omega g_m \triangle f_m \leq C \int_\Omega |g_m||\nabla^2 f_m| \leq C \left(\int_\Omega |g_m|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla^2 f_m|^2\right)^{\frac{1}{2}}$$
$$\to C||f||_2||\nabla^2 f||_2$$

as $k \to \infty$. The left hand side converges to $\int_\Omega |\nabla f|^2$:

$$\left|\int_\Omega \nabla f_m \cdot \nabla g_m - |\nabla f|^2\right| \leq \int_\Omega |\nabla f_m (\nabla g_m - \nabla f) + \nabla f (\nabla f_m - \nabla f)|$$
$$\leq ||\nabla f_m||_2 ||\nabla g_m - \nabla f||_2 + ||\nabla f||_2 ||\nabla f_m - \nabla f||_2 \to 0.$$

In total we have

$$\int_\Omega |\nabla f|^2 \leq C||f||_2||\nabla^2 f||_2.$$

Taking the square root gives the desired estimate.

c) The same calculations as in a) gives

$$\int_\Omega |\nabla f|^{2p} \leq C \int_\Omega |f||\nabla^2 f||\nabla f|^{2(p-1)} \leq C||f||_\infty \int_\Omega |\nabla^2 f||\nabla f|^{2(p-1)}.$$

We use Holders inequality and match the exponents of $|\nabla f|$ and recalling $p \geq 1$:

$$\int_\Omega |\nabla f|^{2p} \leq C||f||_\infty \left(\int_\Omega |\nabla f|^{2p}\right)^{1-\frac{1}{p}} \left(\int_\Omega |\nabla^2 f|^p\right)^{\frac{1}{p}}$$

This gives

$$||\nabla f||_{2p}^2 \leq C||f||_\infty ||\nabla^2 f||_p.$$

Taking the square root gives the estimate.